Advanced Engineering Mathematics

Alan Jeffrey

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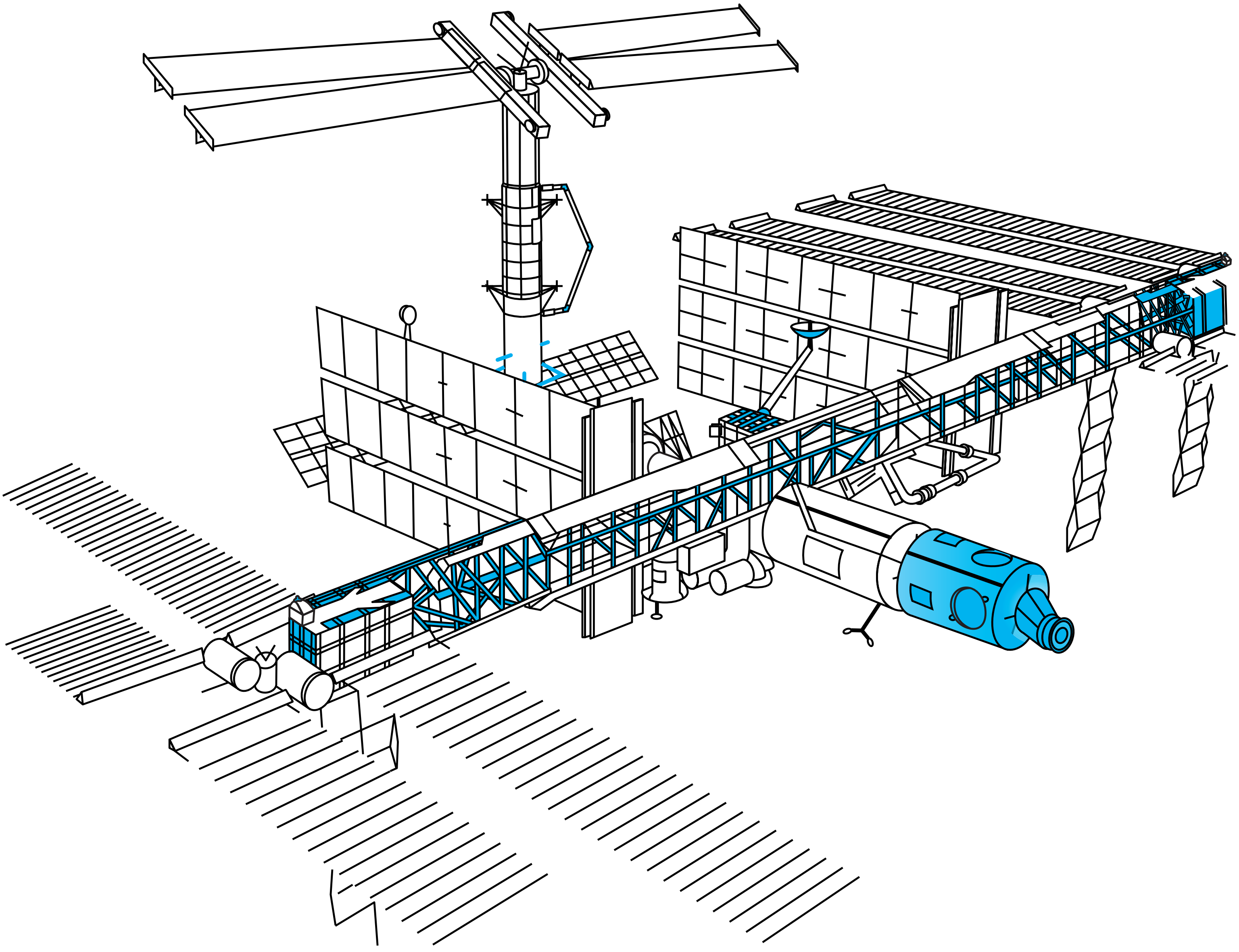

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HARCOURT ACADEMIC PRESS

San Diego San Francisco New York Boston
London Toronto Sydney Tokyo

Sponsoring Editor	Barbara Holland
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Cover Design	Monty Lewis Design
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Front Matter Design	Perspectives
Copyeditor	Kristin Landon
Composition	TechBooks
Printer	RR Donnelley & Sons, Inc.

This book is printed on acid-free paper. ♾

Harcourt/Academic Press
A Harcourt Science and Technology Company
200 Wheeler Road, Burlington, Massachusetts 01803, USA
http://www.harcourt-ap.com

Academic Press
A Harcourt Science and Technology Company
525 B Street, Suite 1900, San Diego, California 92101-4495, USA
http://www.academicpress.com

Academic Press
Harcourt Place, 32 Jamestown Road, London NW1 7BY, UK
http://www.academicpress.com

Library of Congress Catalog Card Number: 00-108262

International Standard Book Number: 0-12-382592-X

PRINTED IN THE UNITED STATES OF AMERICA
01 02 03 04 05 06 DOC 9 8 7 6 5 4 3 2 1

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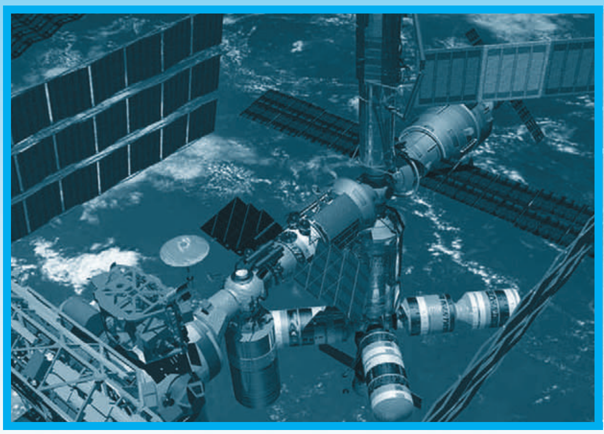

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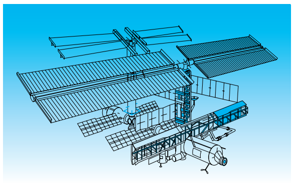

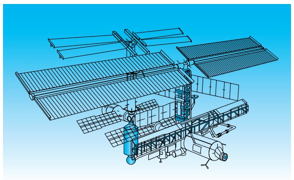

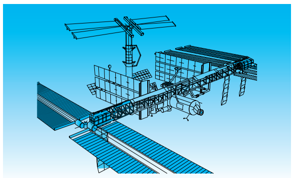

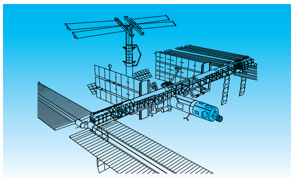

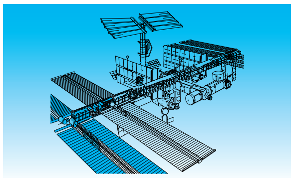

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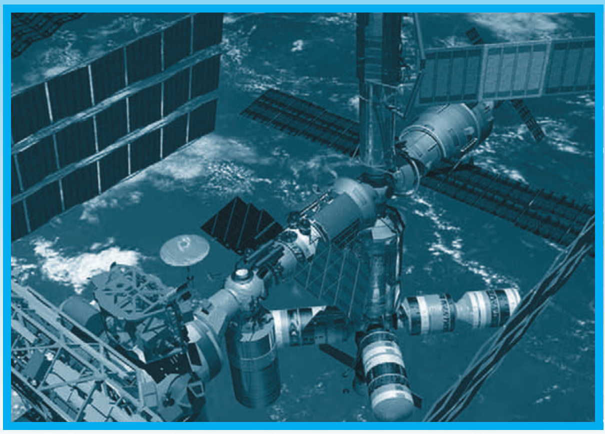

PREFACE

This book has evolved from lectures on engineering mathematics given regularly over many years to students at all levels in the United States, England, and elsewhere. It covers the more advanced aspects of engineering mathematics that are common to all first engineering degrees, and it differs from texts with similar names by the emphasis it places on certain topics, the systematic development of the underlying theory before making applications, and the inclusion of new material. Its special features are as follows.

Prerequisites

The opening chapter, which reviews mathematical prerequisites, serves two purposes. The first is to refresh ideas from previous courses and to provide basic self-contained reference material. The second is to remove from the main body of the text certain elementary material that by tradition is usually reviewed when first used in the text, thereby allowing the development of more advanced ideas to proceed without interruption.

Worked Examples

The numerous worked examples that follow the introduction of each new idea serve in the earlier chapters to illustrate applications that require relatively little background knowledge. The ability to formulate physical problems in mathematical terms is an essential part of all mathematics applications. Although this is not a text on mathematical modeling, where more complicated physical applications are considered, the essential background is first developed to the point at which the physical nature of the problem becomes clear. Some examples, such as the ones involving the determination of the forces acting in the struts of a framed structure, the damping of vibrations caused by a generator and the vibrational modes of clamped membranes, illustrate important mathematical ideas in the context of practical applications. Other examples occur without specific applications and their purpose is to reinforce new mathematical ideas and techniques as they arise.

A different type of example is the one that seeks to determine the height of the tallest flagpole, where the height limitation is due to the phenomenon of

buckling. Although the model used does not give an accurate answer, it provides a typical example of how a mathematical model is constructed. It also illustrates the reasoning used to select a physical solution from a scenario in which other purely mathematical solutions are possible. In addition, the example demonstrates how the choice of a unique physically meaningful solution from a set of mathematically possible ones can sometimes depend on physical considerations that did not enter into the formulation of the original problem.

Exercise Sets

The need for engineering students to have a sound understanding of mathematics is recognized by the systematic development of the underlying theory and the provision of many carefully selected fully worked examples, coupled with their reinforcement through the provision of large sets of exercises at the ends of sections. These sets, to which answers to odd-numbered exercises are listed at the end of the book, contain many routine exercises intended to provide practice when dealing with the various special cases that can arise, and also more challenging exercises, each of which is starred, that extend the subject matter of the text in different ways.

Although many of these exercises can be solved quickly by using standard computer algebra packages, the author believes the fundamental mathematical ideas involved are only properly understood once a significant number of exercises have first been solved by hand. Computer algebra can then be used with advantage to confirm the results, as is required in various exercise sets. Where computer algebra is either required or can be used to advantage, the exercise numbers are in blue. A comparison of computer-based solutions with those obtained by hand not only confirms the correctness of hand calculations, but also serves to illustrate how the method of solution often determines its form, and that transforming one form of solution to another is sometimes difficult. It is the author's belief that only when fundamental ideas are fully understood is it safe to make routine use of computer algebra, or to use a numerical package to solve more complicated problems where the manipulation involved is prohibitive, or where a numerical result may be the only form of solution that is possible.

New Material

Typical of some of the new material to be found in the book is the matrix exponential and its application to the solution of linear systems of ordinary differential equations, and the use of the Green's function. The introductory discussion of the development of discontinuous solutions of first order quasilinear equations, which are essential in the study of supersonic gas flow and in various other physical applications, is also new and is not to be found elsewhere. The account of the Laplace transform contains more detail than usual. While the Laplace transform is applied to standard engineering problems, including

control theory, various nonstandard problems are also considered, such as the solution of a boundary value problem for the equation that describes the bending of a beam and the derivation of the Laplace transform of a function from its differential equation. The chapter on vector integral calculus first derives and then applies two fundamental vector transport theorems that are not found in similar texts, but which are of considerable importance in many branches of engineering.

Series Solutions of Differential Equations

Understanding the derivation of series solutions of ordinary differential equations is often difficult for students. This is recognized by the provision of detailed examples, followed by carefully chosen sets of exercises. The worked examples illustrate all of the special cases that can arise. The chapter then builds on this by deriving the most important properties of Legendre polynomials and Bessel functions, which are essential when solving partial differential equations involving cylindrical and spherical polar coordinates.

Complex Analysis

Because of its importance in so many different applications, the chapters on complex analysis contain more topics than are found in similar texts. In particular, the inclusion of an account of the inversion integral for the Laplace transform makes it possible to introduce transform methods for the solution of problems involving ordinary and partial differential equations for which tables of transform pairs are inadequate. To avoid unnecessary complication, and to restrict the material to a reasonable length, some topics are not developed with full mathematical rigor, though where this occurs the arguments used will suffice for all practical purposes. If required, the account of complex analysis is sufficiently detailed for it to serve as a basis for a single subject course.

Conformal Mapping and Boundary Value Problems

Sufficient information is provided about conformal transformations for them to be used to provide geometrical insight into the solution of some fundamental two-dimensional boundary value problems for the Laplace equation. Physical applications are made to steady-state temperature distributions, electrostatic problems, and fluid mechanics. The conformal mapping chapter also provides a quite different approach to the solution of certain two-dimensional boundary value problems that in the subsequent chapter on partial differential equations are solved by the very different method of separation of variables.

Partial Differential Equations

An understanding of partial differential equations is essential in all branches of engineering, but accounts in engineering mathematics texts often fall short of what is required. This is because of their tendency to focus on the three standard types of linear second order partial differential equations, and their solution by means of separation of variables, to the virtual exclusion of first order equations and the systems from which these fundamental linear second order equations are derived. Often very little is said about the types of boundary and initial conditions that are appropriate for the different types of partial differential equations. Mention is seldom if ever made of the important part played by nonlinearity in first order equations and the way it influences the properties of their solutions. The account given here approaches these matters by starting with first order linear and quasilinear equations, where the way initial and boundary conditions and nonlinearity influence solutions is easily understood. The discussion of the effects of nonlinearity is introduced at a comparatively early stage in the study of partial differential equations because of its importance in subjects like fluid mechanics and chemical engineering. The account of nonlinearity also includes a brief discussion of shock wave solutions that are of fundamental importance in both supersonic gas flow and elsewhere.

Linear and nonlinear wave propagation is examined in some detail because of its considerable practical importance; in addition, the way integral transform methods can be used to solve linear partial differential equations is described. From a rigorous mathematical point of view, the solution of a partial differential equation by the method of separation of variables only yields a formal solution, which only becomes a rigorous solution once the completeness of any set of eigenfunctions that arises has been established. To develop the subject in this manner would take the text far beyond the level for which it is intended and so the completeness of any set of eigenfunctions that occurs will always be assumed. This assumption can be fully justified when applying separation of variables to the applications considered here and also in virtually all other practical cases.

Technology Projects

To encourage the use of technology and computer algebra and to broaden the range of problems that can be considered, technology-based projects have been added wherever appropriate; in addition, standard sets of exercises of a theoretical nature have been included at the ends of sections. These projects are not linked to a particular computer algebra package: Some projects illustrating standard results are intended to make use of simple computer skills while others provide insight into more advanced and physically important theoretical questions. Typical of the projects designed to introduce new ideas are those at the end of the chapter on partial differential equations, which offer a brief introduction to the special nonlinear wave solutions called solitons.

Numerical Mathematics

Although an understanding of basic numerical mathematics is essential for all engineering students, in a book such as this it is impossible to provide a systematic account of this important discipline. The aim of this chapter is to provide a general idea of how to approach and deal with some of the most important and frequently encountered numerical operations, using only basic numerical techniques, and thereafter to encourage the use of standard numerical packages. The routines available in numerical packages are sophisticated, highly optimized and efficient, but the general ideas that are involved are easily understood once the material in the chapter has been assimilated. The accounts that are given here purposely avoid going into great detail as this can be found in the quoted references. However, the chapter does indicate when it is best to use certain types of routine and those circumstances where routines might be inappropriate.

The details of references to literature contained in square brackets at the ends of sections are listed at the back of the book with suggestions for additional reading. An instructor's *Solutions Manual* that gives outline solutions for the technology projects is also available.

Acknowledgments

I wish to express my sincere thanks to the reviewers and accuracy readers, those cited below and many who remain anonymous, whose critical comments and suggestions were so valuable, and also to my many students whose questions when studying the material in this book have contributed so fundamentally to its development. Particular thanks go to:

Chun Liu, Pennsylvania State University
William F. Moss, Clemson University
Donald Hartig, California Polytechnic State University at San Luis Obispo
Howard A. Stone, Harvard University
Donald Estep, Georgia Institute of Technology
Preetham B. Kumar, California State University at Sacramento
Anthony L. Peressini, University of Illinois at Urbana-Champaign
Eutiquio C. Young, Florida State University
Colin H. Marks, University of Maryland
Ronald Jodoin, Rochester Institute of Technology
Edgar Pechlaner, Simon Fraser University
Ronald B. Guenther, Oregon State University
Mattias Kawski, Arizona State University
L. F. Shampine, Southern Methodist University

In conclusion, I also wish to thank my editor, Barbara Holland, for her invaluable help and advice on presentation; Julie Bolduc, senior production editor, for her patience and guidance; Mike Sugarman, for his comments during the early stages of writing; and, finally, Chuck Glaser, for encouraging me to write the book in the first place.

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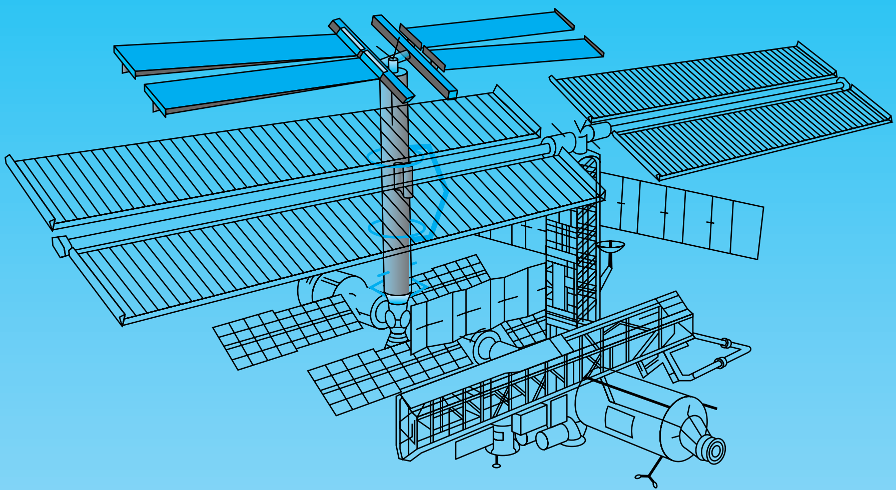
PART ONE

REVIEW MATERIAL



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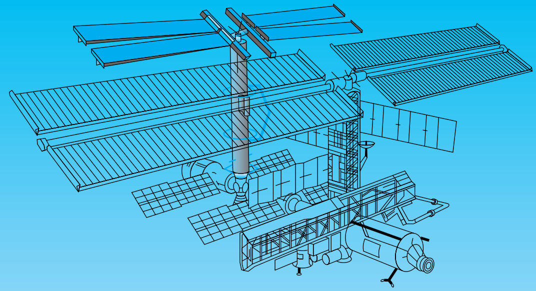
CHAPTER 1



Review of Prerequisites

Every account of advanced engineering mathematics must rely on earlier mathematics courses to provide the necessary background. The essentials are a first course in calculus and some knowledge of elementary algebraic concepts and techniques. The purpose of the present chapter is to review the most important of these ideas that have already been encountered, and to provide for convenient reference results and techniques that can be consulted later, thereby avoiding the need to interrupt the development of subsequent chapters by the inclusion of review material prior to its use.

Some basic mathematical conventions are reviewed in Section 1.1, together with the method of proof by mathematical induction that will be required in later chapters. The essential algebraic operations involving complex numbers are summarized in Section 1.2, the complex plane is introduced in Section 1.3, the modulus and argument representation of complex numbers is reviewed in Section 1.4, and roots of complex numbers are considered in Section 1.5. Some of this material is required throughout the book, though its main use will be in Part 5 when developing the theory of analytic functions.

The use of partial fractions is reviewed in Section 1.6 because of the part they play in Chapter 7 in developing the Laplace transform. As the most basic properties of determinants are often required, the expansion of determinants is summarized in Section 1.7, though a somewhat fuller account of determinants is to be found later in Section 3.3 of Chapter 3.

The related concepts of limit, continuity, and differentiability of functions of one or more independent variables are fundamental to the calculus, and to the use that will be made of them throughout the book, so these ideas are reviewed in Sections 1.8 and 1.9. Tangent line and tangent plane approximations are illustrated in Section 1.10, and improper integrals that play an essential role in the Laplace and Fourier transforms, and also in complex analysis, are discussed in Section 1.11.

The importance of Taylor series expansions of functions involving one or more independent variables is recognized by their inclusion in Section 1.12. A brief mention is also made of the two most frequently used tests for the convergence of series, and of the differentiation and integration of power series that is used in Chapter 8 when considering series solutions of linear ordinary differential equations. These topics are considered again in Part 5 when the theory of analytic functions is developed.

The solution of many problems involving partial differential equations can be simplified by a convenient choice of coordinate system, so Section 1.13 reviews the theorem for the

change of variable in partial differentiation, and describes the cylindrical polar and spherical polar coordinate systems that are the two that occur most frequently in practical problems.

Because of its fundamental importance, the implicit function theorem is stated without proof in Section 1.14, though it is not usually mentioned in first calculus courses.

1.1 Real Numbers, Mathematical Induction, and Mathematical Conventions

Numbers are fundamental to all mathematics, and real numbers are a subset of complex numbers. A real number can be classified as being an **integer**, a **rational** number, or an **irrational** number. From the set of positive and negative integers, and zero, the set of positive integers $1, 2, 3, \ldots$ is called the set of **natural numbers**. The rational numbers are those that can be expressed in the form m/n, where m and n are integers with $n \neq 0$. Irrational numbers such as π, $\sqrt{2}$, and $\sin 2$ are numbers that cannot be expressed in rational form, so, for example, for no integers m and n is it true that $\sqrt{2}$ is equal to m/n. Practical calculations can only be performed using rational numbers, so all irrational numbers that arise must be approximated arbitrarily closely by rational numbers.

Collectively, the sets of integers and rational and irrational numbers form what is called the set of all **real numbers**, and this set is denoted by $\mathbf{R}$. When it is necessary to indicate that an arbitrary number a is a real number a shorthand notation is adopted involving the symbol $\in$, and we will write $a \in \mathbf{R}$. The symbol $\in$ is to be read "belongs to" or, more formally, as "is an element of the set." If a is not a member of set $\mathbf{R}$, the symbol $\in$ is negated by writing $\notin$, and we will write $a \notin \mathbf{R}$ where, of course, the symbol $\notin$ is to be read as "does not belong to," or "is not an element of the set." As real numbers can be identified in a unique manner with points on a line, the set of all real numbers $\mathbf{R}$ is often called the **real line**. The set of all complex numbers $\mathbf{C}$ to which $\mathbf{R}$ belongs will be introduced later.

One of the most important properties of real numbers that distinguishes them from other complex numbers is that they can be arranged in numerical order. This fundamental property is expressed by saying that the real numbers possess the **order property**. This simply means that if $x, y \in \mathbf{R}$, with $x \neq y$, then

$$\text{either } x < y \quad \text{or} \quad x > y,$$

where the symbol $<$ is to be read "is less than" and the symbol $>$ is to be read "is greater than." When the foregoing results are expressed differently, though equivalently, if $x, y \in \mathbf{R}$, with $x \neq y$, then

$$\text{either } x - y < 0 \quad \text{or} \quad x - y > 0.$$

It is the order property that enables the graph of a real function f of a real variable x to be constructed. This follows because once length scales have been chosen for the axes together with a common origin, a real number can be made to correspond to a unique point on an axis. The graph of f follows by plotting all possible points $(x, f(x))$ in the plane, with x measured along one axis and $f(x)$ along the other axis.

absolute value

The **absolute value** $|x|$ of a real number x is defined by the formula

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

This form of definition is in reality a concise way of expressing two separate statements. One statement is obtained by reading $|x|$ with the top condition on the right and the other by reading it with the bottom condition on the right. The absolute value of a real number provides a measure of its magnitude without regard to its sign so, for example, $|3| = 3$, $|-7.41| = 7.41$, and $|0| = 0$.

Sometimes the form of a general mathematical result that only depends on an arbitrary natural number n can be found by experiment or by conjecture, and then the problem that remains is how to prove that the result is either true or false for all n. A typical example is the proposition that the product

$$(1 - 1/4)(1 - 1/9)(1 - 1/16)\ldots[1 - 1/(n+1)^2]$$
$$= (n+2)/(2n+2), \quad \text{for } n = 1, 2, \ldots.$$

This assertion is easily checked for any specific positive integer n, but this does not amount to a proof that the result is true for all natural numbers.

mathematical induction

A powerful method by which such propositions can often be shown to be either true or false involves using a form of argument called **mathematical induction**. This type of proof depends for its success on the order property of numbers and the fact that if n is a natural number, then so also is $n + 1$. The steps involved in an inductive proof can be summarized as follows.

Proof by Mathematical Induction

Let $P(n)$ be a proposition depending on a positive integer n.

STEP 1 Show, if possible, that $P(n)$ is true for some positive integer n_0.

STEP 2 Show, if possible, that if $P(n)$ is true for an arbitrary integer $n = k \geq n_0$, then the proposition $P(k+1)$ follows from proposition $P(k)$.

STEP 3 If Step 2 is true, the fact that $P(n_0)$ is true implies that $P(n_0 + 1)$ is true, and then that $P(n_0 + 2)$ is true, and hence that $P(n)$ is true for all $n \geq n_0$.

STEP 4 If no number $n = n_0$ can be found for which Step 1 is true, or if in Step 2 it can be shown that $P(k)$ does not imply $P(k+1)$, the proposition $P(n)$ is false.

The example that follows is typical of the situation where an inductive proof is used. It arises when determining the nth term in the Maclaurin series for $\sin ax$ that involves finding the nth derivative of $\sin ax$. A result such as this may be found intuitively by inspection of the first few derivatives, though this does not amount to a formal proof that the result is true for all natural numbers n.

EXAMPLE 1.1 Prove by mathematical induction that

$$d^n/dx^n[\sin ax] = a^n \sin(ax + n\pi/2), \quad \text{for } n = 1, 2, \ldots.$$

Solution The proposition $P(n)$ is that

$$d^n/dx^n[\sin ax] = a^n \sin(ax + n\pi/2), \quad \text{for } n = 1, 2, \ldots.$$

STEP 1 Differentiation gives

$$d/dx[\sin ax] = a \cos ax,$$

but setting $n = 1$ in $P(n)$ leads to the result

$$d/dx[\sin ax] = a\sin(ax + \pi/2) = a\cos ax,$$

showing that proposition $P(n)$ is true for $n = 1$ (so in this case $n_0 = 1$).

STEP 2 Assuming $P(k)$ to be true for $k > 1$, differentiation gives

$$d/dx\{d^k/dx^k[\sin ax]\} = d/dx[a^k \sin(ax + k\pi/2)],$$

so

$$d^{k+1}/dx^{k+1}[\sin ax] = a^{k+1}\cos(ax + k\pi/2).$$

However, replacing k by $k+1$ in $P(k)$ gives

$$\begin{aligned} d^{k+1}/dx^{k+1}[\sin ax] &= a^{k+1}\sin[ax + (k+1)\pi/2] \\ &= a^{k+1}\sin[(ax + k\pi/2) + \pi/2] \\ &= a^{k+1}\cos(ax + k\pi/2), \end{aligned}$$

showing, as required, that proposition $P(k)$ implies proposition $P(k+1)$, so Step 2 is true.

STEP 3 As $P(n)$ is true for $n = 1$, and $P(k)$ implies $P(k+1)$, it follows that the result is true for $n = 1, 2, \ldots$ and the proof is complete. ■

The **binomial theorem** finds applications throughout mathematics at all levels, so we quote it first when the exponent n is a positive integer, and then in its more general form when the exponent α involved is any real number.

Binomial theorem when n is a positive integer

If a, b are real numbers and n is a positive integer, then

$$\begin{aligned} (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 \\ + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n, \end{aligned}$$

binomial coefficient

or more concisely in terms of the **binomial coefficient**

$$\binom{n}{r} = \frac{n!}{(n-r)!r!},$$

we have

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r,$$

where $m!$ is the factorial function defined as $m! = 1 \cdot 2 \cdot 3 \cdots m$ with $m > 0$ an integer, and $0!$ is defined as $0! = 1$. It follows at once that

$$\binom{n}{0} = \binom{n}{n} = 1.$$

The binomial theorem involving the expression $(a+b)^\alpha$, where a and b are real numbers with $|b/a| < 1$ and α is an arbitrary real number takes the following form.

General form of the binomial theorem when α is an arbitrary real number

If a and b are real numbers such that $|b/a| < 1$ and α is an arbitrary real number, then

$$(a+b)^\alpha = a^\alpha\left(1+\frac{b}{a}\right)^\alpha = a^\alpha\left(1+\frac{\alpha}{1!}\left(\frac{b}{a}\right)+\frac{\alpha(\alpha-1)}{2!}\left(\frac{b}{a}\right)^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\left(\frac{b}{a}\right)^3+\cdots\right).$$

The series on the right only terminates after a finite number of terms if α is a positive integer, in which case the result reduces to the one just given. If α is a negative integer, or a nonintegral real number, the expression on the right becomes an infinite series that diverges if $|b/a| > 1$.

EXAMPLE 1.2

Expand $(3+x)^{-1/2}$ by the binomial theorem, stating for what values of x the series converges.

Solution Setting $b/a = \frac{1}{3}x$ in the general form of the binomial theorem gives

$$(3+x)^{-1/2} = 3^{-1/2}\left(1+\frac{1}{3}x\right)^{-1/2} = \frac{1}{\sqrt{3}}\left(1-\frac{1}{6}x+\frac{1}{24}x^2-\frac{5}{432}x^3+\cdots\right).$$

The series only converges if $|\frac{1}{3}x| < 1$, and so it is convergent provided $|x| < 3$. ■

Some standard mathematical conventions

Use of combinations of the $\pm$ and $\mp$ signs

The occurrence of two or more of the symbols $\pm$ and $\mp$ in an expression is to be taken to imply two separate results, the first obtained by taking the upper signs and the second by taking the lower signs. Thus, the expression $a \pm b\sin\theta \mp c\cos\theta$ is an abbreviation for the two separate expressions

$$a + b\sin\theta - c\cos\theta \quad \text{and} \quad a - b\sin\theta + c\cos\theta.$$

Multi-statements

multi-statement

When a function is defined sectionally on n different intervals of the real line, instead of formulating n separate definitions these are usually simplified by being combined into what can be considered to be a single **multi-statement**. The following example is typical of a multi-statement:

$$f(x) = \begin{cases} \sin x, & x < \pi \\ 0, & \pi \le x \le 3\pi/2 \\ -1, & x > 3\pi/2. \end{cases}$$

It is, in fact, three statements. The first is obtained by reading $f(x)$ in conjunction with the top line on the right, the second by reading it in conjunction with the second line on the right, and the third by reading it in conjunction with the third line on the right. An example of a multi-statement has already been encountered in the definition of the absolute value $|x|$ of a number x. Frequent use of multi-statements will be made in Chapter 9 on Fourier series, and elsewhere.

Polynomials

polynomials

A **polynomial** is an expression of the form $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. The integer n is called the **degree** of the polynomial, and the numbers a_i are called its **coefficients**. The **fundamental theorem of algebra** that is proved in Chapter 14 asserts that $P(x) = 0$ has n **roots** that may be either real or complex, though some of them may be repeated. ($a_0 \neq 0$ is assumed.)

Notation for ordinary and partial derivatives

If $f(x)$ is an n times differentiable function then $f^{(n)}(x)$ will, on occasion, be used to signify $d^n f/dx^n$, so that

$$f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

suffix notation for partial derivatives

If $f(x, y)$ is a suitably differentiable function of x and y, a concise notation used to signify partial differentiation involves using suffixes, so that

$$f_x = \frac{\partial f}{\partial x},\ f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x},\ f_{yy} = \frac{\partial^2 f}{\partial y^2}, \ldots,$$

with similar results when f is a function of more than two independent variables.

Inverse trigonometric functions

The periodicity of the real variable trigonometric sine, cosine, and tangent functions means that the corresponding general inverse trigonometric functions are *many valued*. So, for example, if $y = \sin x$ and we ask for what values of x is $y = 1/\sqrt{2}$, we find this is true for $x = \pi/4 \pm 2n\pi$ and $x = 3\pi/4 \pm 2n\pi$ for $n = 0, 1, 2, \ldots$. To overcome this ambiguity, we introduce the *single valued* inverses, denoted respectively by $x = \text{Arcsin } y$, $x = \text{Arccos } y$, and $x = \text{Arctan } y$ by restricting the domain and range of the sine, cosine, and tangent functions to one where they are either strictly increasing or strictly decreasing functions, because then one value of x corresponds to one value of y and, conversely, one value of y corresponds to one value of x.

In the case of the function $y = \sin x$, by restricting the argument x to the interval $-\pi/2 \leq x \leq \pi/2$ the function becomes a strictly increasing function of x. The corresponding single valued inverse function is denoted by $x = \text{Arcsin } y$, where y is a number in the domain of definition $[-1, 1]$ of the Arcsine function and x is a number in its range $[-\pi/2, \pi/2]$. Similarly, when considering the function $y = \cos x$, the argument is restricted to $0 \leq x \leq \pi$ to make $\cos x$ a strictly decreasing function of x. The corresponding single valued inverse function is denoted by $x = \text{Arccos } y$, where y is a number in the domain of definition $[-1, 1]$ of the Arccosine function and x is a number in its range $[0, \pi]$. Finally, in the case of the function $y = \tan x$, restricting

the argument to the interval $-\pi/2 < x < \pi/2$ makes the tangent function a strictly increasing function of x. The corresponding single valued inverse function is denoted by $x = \text{Arctan}\, y$ where y is a number in the domain of definition $(-\infty, \infty)$ of the Arctangent function and x is a number in its range $(-\pi/2, \pi/2)$.

As the inverse trigonometric functions are important in their own right, the variables x and y in the preceding definitions are interchanged to allow consideration of the inverse functions $y = \text{Arcsin}\, x$, $y = \text{Arccos}\, x$, and $y = \text{Arctan}\, x$, so that now x is the independent variable and y is the dependent variable.

With this interchange of variables the expression $y = \arcsin x$ will be used to refer to any single valued inverse function with the *same* domain of definition as $\text{Arcsin}\, x$, but with a *different* range. Similar definitions apply to the functions $y = \arccos x$ and $y = \arctan x$.

Double summations

An expression involving a double summation like

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin mx \sin ny,$$

double summation

means sum the terms $a_{mn} \sin mx \sin ny$ over all possible values of m and n, so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin mx \sin ny = a_{11} \sin x \sin y + a_{12} \sin x \sin 2y + a_{21} \sin 2x \sin y + a_{22} \sin 2x \sin 2y + \cdots .$$

A more concise notation also in use involves writing the double summation as

$$\sum_{m=1,n=1}^{\infty} a_{mn} \sin mx \sin nx.$$

The signum function

signum function

The **signum function**, usually written $\text{sign}(x)$, and sometimes $\text{sgn}(x)$, is defined as

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

We have, for example, $\text{sign}(\cos x) = 1$ for $0 < x < \pi/2$, and $\text{sign}(\cos x) = -1$ for $\pi/2 < x < \pi$ or, equivalently,

$$\text{sign}(\cos x) = \begin{cases} 1, & 0 < x < \frac{1}{2}\pi \\ -1, & \frac{1}{2}\pi < x < \pi. \end{cases}$$

Products

Let $\{u_k\}_{k=1}^{n}$ be a sequence of numbers or functions $u_1, u_2, \ldots$; then the product of the n members of this sequence is denoted by $\prod_{k=1}^{n} u_k$, so that

$$\prod_{k=1}^{n} u_k = u_1 u_2 \cdots u_n.$$

infinite product

When the sequence is infinite,

$$\lim_{n\to\infty} \prod_{k=1}^{n} u_k = \prod_{k=1}^{\infty} u_k$$

is called an **infinite product** involving the sequence $\{u_k\}$. Typical examples of infinite products are

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)=\frac{1}{2} \quad \text{and} \quad \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)=\frac{\sin x}{x}.$$

More background information and examples can be found in the appropriate sections in any of references [1.1], [1.2], and [1.5].

Logarithmic functions

the functions ln and Log

The notation $\ln x$ is used to denote the **natural logarithm** of a real number x, that is, the logarithm of x to the base e, and in some books this is written $\log_e x$. In this book logarithms to the base 10 are not used, and when working with functions of a complex variable the notation $\mathrm{Log}\, z$, with $z = re^{i\theta}$ means $\mathrm{Log}\, z = \ln r + i\theta$.

EXERCISES 1.1

1. Prove that if $a > 0, b > 0$, then $a/\sqrt{b} + b/\sqrt{a} \geq \sqrt{a} + \sqrt{b}$.

Prove Exercises 2 through 6 by mathematical induction.

2. $\sum_{k=0}^{n-1}(a + kd) = (n/2)[2a + (n-1)d]$ (sum of an arithmetic series).
3. $\sum_{k=0}^{n-1} r^k = (1 - r^n)/(1 - r) \quad (r \neq 1)$ (sum of a geometric series).
4. $\sum_{k=1}^{n} k^2 = (1/6)n(n+1)(2n+1)$ (sum of squares).
5. $d^n/dx^n[\cos ax] = a^n \cos(ax + n\pi/2)$, with n a natural number.
6. $d^n/dx^n[\ln(1 + x)] = (-1)^{n+1}(n-1)!/(1 + x)^n$, with n a natural number.
7. Use the binomial theorem to expand $(3 + 2x)^4$.
8. Use the binomial theorem and multiplication to expand $(1 - x^2)(2 + 3x)^3$.

In Exercises 9 through 12 find the first four terms of the binomial expansion of the function and state conditions for the convergence of the series.

9. $(3 + 2x)^{-2}$.
10. $(2 - x^2)^{1/3}$.
11. $(4 + 2x^2)^{-1/2}$.
12. $(1 - 3x^2)^{3/4}$.

1.2 Complex Numbers

Mathematical operations can lead to numbers that do not belong to the real number system $\mathbf{R}$ introduced in Section 1.1. In the simplest case this occurs when finding the roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad \text{with } a, b, c \in \mathbf{R}, a \neq 0$$

by means of the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

discriminant of a quadratic

The **discriminant** of the equation is $b^2 - 4ac$, and if $b^2 - 4ac < 0$ the formula involves the square root of a negative real number; so, if the formula is to have meaning, numbers must be allowed that lie outside the real number system.

The inadequacy of the real number system when considering different mathematical operations can be illustrated in other ways by asking, for example, how to find the three roots that are expected of a third degree algebraic equation as

simple as $x^3 - 1 = 0$, where only the real root 1 can be found using $y = x^3 - 1$, or by seeking to give meaning to $\ln(-1)$, both of which questions will arise later.

Difficulties such as these can all be overcome if the real number system is extended by introducing the **imaginary unit** i defined as

$$i^2 = -1,$$

so expressions like $\sqrt{(-k^2)}$ where k a positive real number may be written $\sqrt{(-1)}\sqrt{(k^2)} = \pm ik$. Notice that as the real number k only *scales* the imaginary unit i, it is immaterial whether the result is written as ik or as ki.

The extension to the real number system that is required to resolve problems of the type just illustrated involves the introduction of **complex numbers**, denoted collectively by **C**, in which the general complex number, usually denoted by z, has the form

$$z = \alpha + i\beta, \quad \text{with } \alpha, \beta \text{ real numbers.}$$

real and imaginary part notation

The real number α is called the **real part** of the complex number z, and the real number β is called its **imaginary part**. When these need to be identified separately, we write

$$\text{Re}\{z\} = \alpha \quad \text{and} \quad \text{Im}\{z\} = \beta,$$

so if $z = 3 - 7i$, $\text{Re}\{z\} = 3$ and $\text{Im}\{z\} = -7$.

If $\text{Im}\{z\} = \beta = 0$ the complex number z reduces to a real number, and if $\text{Re}\{z\} = \alpha = 0$ it becomes a purely imaginary number, so, for example, $z = 5i$ is a purely imaginary number. When a complex number z is considered as a variable it is usual to write it as

$$z = x + iy,$$

where x and y are now real variables. If it is necessary to indicate that z is a general complex number we write $z \in \mathbf{C}$.

When solving the quadratic equation $az^2 + bz + c = 0$ with a, b, and c real numbers and a discriminant $b^2 - 4ac < 0$, by setting $4ac - b^2 = k^2$ in the quadratic formula, with $k > 0$, the two roots z_1 and z_2 are given by the complex numbers

$$z_1 = -(b/2a) + i(k/2a) \quad \text{and} \quad z_2 = -(b/2a) - i(k/2a).$$

Algebraic rules for complex numbers

Let the complex numbers z_1 and z_2 be defined as

$$z_1 = a + ib \quad \text{and} \quad z_2 = c + id,$$

with a, b, c, and d arbitrary real numbers. Then the following rules govern the arithmetic manipulation of complex numbers.

Equality of complex numbers

The complex numbers z_1 and z_2 are **equal**, written $z_1 = z_2$ if, and only if, $\text{Re}\{z_1\} = \text{Re}\{z_2\}$ and $\text{Im}\{z_1\} = \text{Im}\{z_2\}$. So $a + ib = c + id$ if, and only if,

$$a = c \quad \text{and} \quad b = d.$$

EXAMPLE 1.3

(a) $3 - 9i = 3 + bi$ if, and only if, $b = -9$.

(b) If $u = -2 + 5i$, $v = 3 + 5i$, $w = a + 5i$, then
$u = w$ if, and only if, $a = -2$ but $u \neq v$, and
$v = w$ if, and only if, $a = 3$. ■

Zero complex number

The **zero** complex number, also called the **null** complex number, is the number $0 + 0i$ that, for simplicity, is usually written as an ordinary zero 0.

EXAMPLE 1.4

If $a + ib = 0$, then $a = 0$ and $b = 0$. ■

Addition and subtraction of complex numbers

The **addition (sum)** and **subtraction (difference)** of the complex numbers z_1 and z_2 is defined as

$$z_1 + z_2 = \mathrm{Re}\{z_1\} + \mathrm{Re}\{z_2\} + i[\mathrm{Im}\{z_1\} + \mathrm{Im}\{z_2\}]$$

and

$$z_1 - z_2 = \mathrm{Re}\{z_1\} - \mathrm{Re}\{z_2\} + i[\mathrm{Im}\{z_1\} - \mathrm{Im}\{z_2\}].$$

So, if $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned} z_1 + z_2 &= (a + ib) + (c + id) \\ &= (a + c) + i(b + d), \end{aligned}$$

and

$$\begin{aligned} z_1 - z_2 &= (a + ib) - (c + id) \\ &= (a - c) + i(b - d). \end{aligned}$$

EXAMPLE 1.5

If $z_1 = 3 + 7i$ and $z_2 = 3 + 2i$, then the sum

$$z_1 + z_2 = (3 + 3) + (7 + 2)i = 6 + 9i,$$

and the difference

$$z_1 - z_2 = (3 - 3) + (7 - 2)i = 5i.$$ ■

Multiplication of complex numbers

The **multiplication (product)** of the two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is defined by the rule

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

An immediate consequence of this definition is that if k is a real number, then $kz_1 = k(a + ib) = ka + ikb$. This operation involving multiplication of a complex

number by a real number is called *scaling* a complex number. Thus, if $z_1 = 3 + 7i$ and $z_2 = 3 + 2i$, then $2z_1 - 3z_2 = (6 + 14i) - (9 + 6i) = -3 + 8i$.

In particular, if $z = a + ib$, then $-z = (-1)z = -a - ib$. This is as would be expected, because it leads to the result $z - z = 0$.

In practice, instead of using this formal definition of multiplication, it is more convenient to perform multiplication of complex numbers by multiplying the bracketed quantities in the usual algebraic manner, replacing every product i^2 by -1, and then combining separately the real and imaginary terms to arrive at the required product.

EXAMPLE 1.6

(a) $5i(-4 + 3i) = -15 - 20i$.

(b) $(3 - 2i)(-1 + 4i)(1 + i) = (-3 + 12i + 2i - 8i^2)(1 + i)$

$$= [(-3 + 8) + (12 + 2)i](1 + i) = (5 + 14i)(1 + i)$$

$$= 5 + 14i + 5i + 14i^2 = (5 - 14) + (5 + 14)i = -9 + 19i.$$ ■

Complex conjugate

If $z = a + ib$, then the **complex conjugate** of z, usually denoted by $\overline{z}$ and read "z bar," is defined as $\overline{z} = a - ib$. It follows directly that

$$\overline{(\overline{z})} = z \quad \text{and} \quad z\overline{z} = a^2 + b^2.$$

In words, the complex conjugate operation has the property that taking the complex conjugate of a complex conjugate returns the original complex number, whereas the product of a complex number and its complex conjugate always yields a real number.

If $z = a + ib$, then adding and subtracting z and $\overline{z}$ gives the useful results

$$z + \overline{z} = 2\mathrm{Re}\{z\} = 2a \quad \text{and} \quad z - \overline{z} = 2i\ \mathrm{Im}\{z\} = 2ib.$$

These can be written in the equivalent form

$$\mathrm{Re}\{z\} = a = \frac{1}{2}(z + \overline{z}) \quad \text{and} \quad \mathrm{Im}\{z\} = b = \frac{1}{2i}(z - \overline{z}).$$

Quotient (division) of complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$. Then the **quotient** z_1/z_2 is defined as

$$\frac{z_1}{z_2} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2},\ z_2 \neq 0.$$

In practice, division of complex numbers is not carried out using this definition. Instead, the quotient is written in the form

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z}_2}{z_2 \overline{z}_2},$$

where the denominator is now seen to be a real number. The quotient is then found by multiplying out and simplifying the numerator in the usual manner and dividing the real and imaginary parts of the numerator by the real number $z_2 \overline{z}_2$.

EXAMPLE 1.7 Find z_1/z_2 given that $z_1 = (3 + 2i)$ and $z_2 = 1 + 3i$.

Solution

$$\frac{3+2i}{1+3i} = \frac{(3+2i)(1-3i)}{(1+3i)(1-3i)} = \frac{3 - 9i + 2i - 6i^2}{10} = \frac{9}{10} - \frac{7i}{10}.$$

■

Modulus of a complex number

The **modulus** of the complex number $z = a + ib$ denoted by $|z|$, and also called its **magnitude**, is defined as

$$|z| = (a^2 + b^2)^{1/2} = (z\overline{z})^{1/2}.$$

It follows directly from the definitions of the modulus and division that

$$|z| = |\overline{z}| = (a^2 + b^2)^{1/2},$$

and

$$z_1/z_2 = z_1 \overline{z}_2 / |z_2|^2.$$

EXAMPLE 1.8 If $z = 3 + 7i$, then $|z| = |3 + 7i| = (3^2 + 7^2)^{1/2} = \sqrt{58}$. ■

It is seen that the foregoing rules for the arithmetic manipulation of complex numbers reduce to the ordinary arithmetic rules for the algebraic manipulation of real numbers when all the complex numbers involved are real numbers. Complex numbers are the most general numbers that need to be used in mathematics, and they contain the real numbers as a special case. There is, however, a fundamental difference between real and complex numbers to which attention will be drawn after their common properties have been listed.

Properties shared by real and complex numbers

Let z, u, and w be arbitrary real or complex numbers. Then the following properties are true:

1. $z + u = u + z.$ This means that the order in which complex numbers are added does not affect their sum.
2. $zu = uz.$ This means that the order in which complex numbers are multiplied does not affect their product.

3. $(z+u)+w = z+(u+w)$. This means that the order in which brackets are inserted into a sum of finitely many complex numbers does not affect the sum.
4. $z(uw) = (zu)w$. This means that the terms in a product of complex numbers may be grouped and multiplied in any order without affecting the resulting product.
5. $z(u+w) = zu+zw$. This means that the product of z and a sum of complex numbers equals the sum of the products of z and the individual complex numbers involved in the sum.
6. $z+0 = 0+z = z$. This result means that the addition of zero to any complex number leaves it unchanged.
7. $z \cdot 1 = 1 \cdot z = z$. This result means that multiplication of any complex number by unity leaves the complex number unchanged.

Despite the properties common to real and complex numbers just listed, there remains a fundamental difference because, unlike real numbers, complex numbers have *no* natural order. So if z_1 and z_2 are any complex numbers, a statement such as $z_1 < z_2$ has no meaning.

EXERCISES 1.2

Find the roots of the equations in Exercises 1 through 6.

1. $z^2+z+1=0$.
2. $2z^2+5z+4=0$.
3. $z^2+z+6=0$.
4. $3z^2+2z+1=0$.
5. $3z^2+3z+1=0$.
6. $2z^2-2z+3=0$.
7. Given that $z=1$ is a root, find the other two roots of $2z^3-z^2+3z-4=0$.
8. Given that $z=-2$ is a root, find the other two roots of $4z^3+11z^2+10z+8=0$.
9. Given $u=4-2i$, $v=3-4i$, $w=-5i$ and $a+ib=(u+iv)w$, find a and b.
10. Given $u=-4+3i$, $v=2+4i$, and $a+ib=uv^2$, find a and b.
11. Given $u=2+3i$, $v=1-2i$, $w=-3-6i$, find $|u+v|$, $u+2v$, $u-3v+2w$, uv, uvw, $|u/v|$, v/w.
12. Given $u=1+3i$, $v=2-i$, $w=-3+4i$, find uv/w, uw/v and $|v|w/u$.

1.3 The Complex Plane

cartesian representation of z

Complex numbers can be represented geometrically either as *points*, or as *directed line segments* (*vectors*), in the **complex plane**. The complex plane is also called the **z-plane** because of the representation of complex numbers in the form $z = x+iy$. Both of these representations are accomplished by using rectangular cartesian coordinates and plotting the complex number $z = a+ib$ as the point (a, b) in the plane, so the x-coordinate of z is $a = \text{Re}\{z\}$ and its y-coordinate is $b = \text{Im}\{z\}$. Because of this geometrical representation, a complex number written in the form $z = a+ib$ is said to be expressed in **cartesian form**. To acknowledge the Swiss amateur mathematician Jean-Robert Argand, who introduced the concept of the complex plane in 1806, and who by profession was a bookkeeper, this representation is also called the **Argand diagram**.

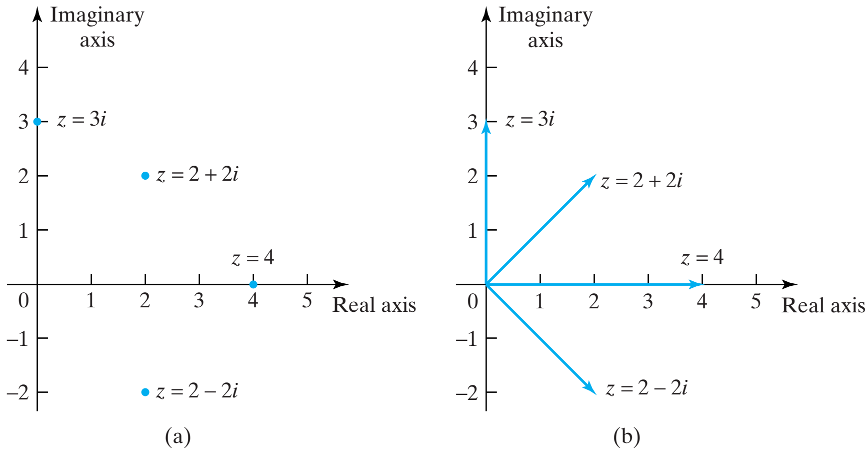


FIGURE 1.1 (a) Complex numbers as points. (b) Complex numbers as vectors.

For obvious reasons, the x-axis is called the **real axis** and the y-axis the **imaginary axis**. Purely real numbers are represented by points on the real axis and purely imaginary ones by points on the imaginary axis. Examples of the representation of typical points in the complex plane are given in Fig. 1.1a, where the numbers $4, 3i, 2 + 2i$, and $2 - 2i$ are plotted as points. These same complex numbers are shown again in Fig. 1.1b as directed line segments drawn from the origin (vectors). The arrow shows the *sense* along the line, that is, the direction from the origin to the tip of the vector representing the complex number. It can be seen from both figures that, when represented in the complex plane, a complex number and its complex conjugate (in this case $2 + 2i$ and $2 - 2i$) lie symmetrically above and below the real axis. Another way of expressing this result is by saying that a complex number and its complex conjugate appear as **reflections** of each other in the real axis, which acts like a mirror.

The addition and subtraction of two complex numbers have convenient geometrical interpretations that follow from the definitions given in Section 1.2. When complex numbers are added, their respective real and imaginary parts are added, whereas when they are subtracted, their respective real and imaginary parts are subtracted. This leads at once to the **triangle law** for addition illustrated in Fig. 1.2a, in which the directed line segment (vector) representing z_2 is translated without rotation or change of scale, to bring its base (the end opposite to the arrow) into coincidence with the tip of the directed line element representing z_1 (the end at which the arrow is located). The sum $z_1 + z_2$ of the two complex numbers is then represented by the directed line segment from the base of the line segment representing z_1 to the tip of the newly positioned line segment representing z_2.

triangle and parallelogram laws

The name *triangle law* comes from the triangle that is constructed in the complex plane during this geometrical process of addition. Notice that an immediate consequence of this law is that addition is *commutative*, because both $z_1 + z_2$ and $z_2 + z_1$ are seen to lead to the same directed line segment in the complex plane. For this reason the addition of complex numbers is also said to obey the **parallelogram law** for addition, because the commutative property generates the parallelogram shown in Fig. 1.2a.

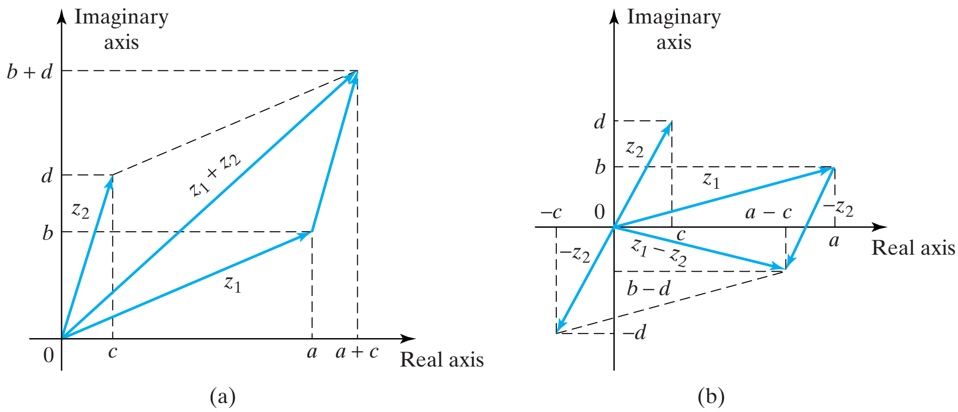


FIGURE 1.2 Addition and subtraction of complex numbers using the triangle/parallelogram law.

The geometrical interpretation of the subtraction of z_2 from z_1 follows similarly by adding to z_1 the directed line segment $-z_2$ that is obtained by reversing of the sense (arrow) along z_2, as shown in Fig. 1.2b.

It is an elementary fact from Euclidean geometry that the sum of the lengths of the two sides $|u|$ and $|v|$ of the triangle in Fig. 1.3 is greater than or equal to the length of the hypotenuse $|u+v|$, so from geometrical considerations we can write

$$|u+v| \leq |u| + |v|.$$

triangle inequality

This result involving the moduli of the complex numbers u and v is called the **triangle inequality** for complex numbers, and it has many applications.

An algebraic proof of the triangle inequality proceeds as follows:

$$\begin{aligned} |u+v|^2 &= (u+v)\overline{(u+v)} = u\bar{u} + v\bar{u} + u\bar{v} + v\bar{v} \\ &= |u|^2 + |v|^2 + (u\bar{v} + \overline{u\bar{v}}) \leq |u^2| + |v^2| + 2|u\bar{v}| \\ &= (|u| + |v|)^2. \end{aligned}$$

The required result now follows from taking the positive square root.

A similar argument, the proof of which is left as an exercise, can be used to show that $||u| - |v|| \leq |u+v|$, so when combined with the triangle inequality we have

$$||u| - |v|| \leq |u+v| \leq |u| + |v|.$$

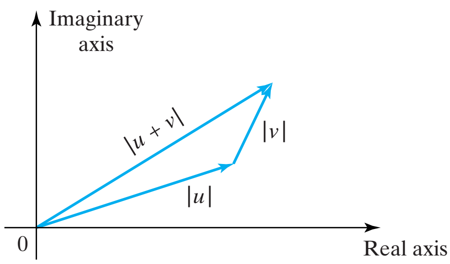


FIGURE 1.3 The triangle inequality.

EXERCISES 1.3

In Exercises 1 through 8 use the parallelogram law to form the sum and difference of the given complex numbers and then verify the results by direct addition and subtraction.

1. $u = 2 + 3i$, $v = 1 - 2i$.
2. $u = 4 + 7i$, $v = -2 - 3i$.
3. $u = -3$, $v = -3 - 4i$.
4. $u = 4 + 3i$, $v = 3 + 4i$.
5. $u = 3 + 6i$, $v = -4 + 2i$.
6. $u = -3 + 2i$, $v = 6i$.
7. $u = -4 + 2i$, $v = -4 - 10i$.
8. $u = 4 + 7i$, $v = -3 + 5i$.

In Exercises 9 through 11 use the parallelogram law to verify the triangle inequality $|u + v| \le |u| + |v|$ for the given complex numbers u and v.

9. $u = -4 + 2i$, $v = 3 + 5i$.
10. $u = 2 + 5i$, $v = 3 - 2i$.
11. $u = -3 + 5i$, $v = 2 + 6i$.

1.4 Modulus and Argument Representation of Complex Numbers

polar representation of z

When representing $z = x + iy$ in the complex plane by a point P with coordinates (x, y), a natural alternative to the cartesian representation is to give the *polar coordinates* (r, θ) of P. This polar representation of z is shown in Fig. 1.4, where

$$OP = r = |z| = (x^2 + y^2)^{1/2} \quad \text{and} \quad \tan\theta = y/x. \tag{1}$$

The radial distance OP is the **modulus** of z, so $r = |z|$, and the angle θ measured counterclockwise from the positive real axis is called the **argument** of z. Because of this, a complex number expressed in terms of the polar coordinates (r, θ) is said to be in **modulus–argument** form. The argument θ is indeterminate up to a multiple of 2π, because the polar coordinates (r, θ), and $(r, \theta + 2k\pi)$, with $k = \pm1, \pm2, \ldots,$ identify the *same* point P. By convention, the the angle θ is called the **principal value** of the argument of z when it lies in the interval $-\pi < \theta \le \pi$. To distinguish the principal value of the argument from all of its other values, we write

$$\operatorname{Arg} z = \theta, \quad \text{when } -\pi < \theta \le \pi. \tag{2}$$

The values of the argument of z that differ from this value of θ by a multiple of 2π are denoted by $\arg z$, so that

$$\arg z = \theta + 2k\pi, \quad \text{with } k = \pm1, \pm2, \ldots. \tag{3}$$

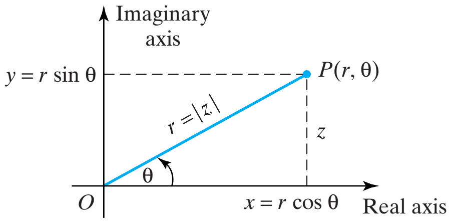


FIGURE 1.4 The complex plane and the (r, θ) representation of z.

The significance of the multivalued nature of arg z will become apparent later when the roots of complex numbers are determined.

The connection between the cartesian coordinates (x, y) and the polar coordinates (r, θ) of the point P corresponding to $z = x + iy$ is easily seen to be given by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

This leads immediately to the representation of $z = x + iy$ in the alternative *modulus–argument form*

modulus–argument representation of z

$$z = r(\cos\theta + i\sin\theta). \tag{4}$$

A routine calculation using elementary trigonometric identities shows that

$$(\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta).$$

An inductive argument using the above result as its first step then establishes the following simple but important theorem.

THEOREM 1.1

De Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta), \quad \text{for } n \text{ a natural number.}$$ ■

EXAMPLE 1.9

Use de Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Solution The result is obtained by first setting $n = 4$ in de Moivre's theorem and expanding $(\cos\theta + i\sin\theta)^4$ to obtain

$$\cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta = \cos 4\theta + i\sin 4\theta.$$

Equating the respective real and imaginary parts on either side of this identity gives the required results

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

and

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$$ ■

As the complex number $z = \cos\theta + i\sin\theta$ has unit modulus, it follows that all numbers of this form lie on the unit circle (a circle of radius 1) centered on the origin, as shown in Fig. 1.5.

Using (5), we see that if $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta), \text{ for } n \text{ a natural number.} \tag{5}$$

The relationship between e^θ, $\sin\theta$, and $\cos\theta$ can be seen from the following well-known series expansions of the functions

$$e^\theta = \sum_{n=0}^{\infty}\frac{\theta^n}{n!} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \cdots;$$

$$\sin\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots;$$

$$\cos\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots.$$

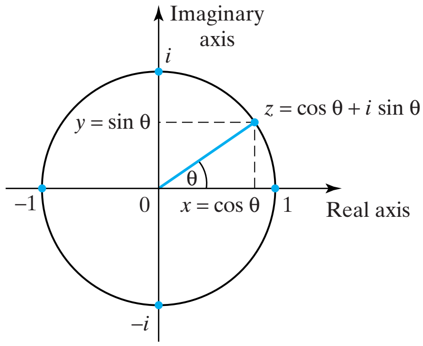


FIGURE 1.5 Point $z = \cos\theta + i\sin\theta$ on the unit circle centered on the origin.

By making a formal power series expansion of the function $e^{i\theta}$, simplifying powers of i, grouping together the real and imaginary terms, and using the series representations for $\cos\theta$ and $\sin\theta$, we arrive at what is called the real variable form of the

Euler formula

Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \text{for any real } \theta. \tag{6}$$

This immediately implies that if $z = re^{i\theta}$, then

$$z^{\alpha} = r^{\alpha}e^{i\alpha\theta}, \quad \text{for any real } \alpha. \tag{7}$$

When θ is restricted to the interval $-\pi < \theta \le \pi$, formula (6) leads to the useful results

$$1 = e^{i0}, \quad i = e^{i\pi/2}, \quad -1 = e^{i\pi}, \quad -i = e^{-i\pi/2}$$

and, in particular, to

$$1 = e^{2k\pi i} \quad \text{for } k = 0, \pm 1, \pm 2, \ldots.$$

The Euler form for complex numbers makes their multiplication and division very simple. To see this we set $z_1 = r_1e^{i\alpha}$ and $z_2 = r_2e^{i\beta}$ and then use the results

$$z_1z_2 = r_1r_2e^{i(\alpha+\beta)} \quad \text{and} \quad z_1/z_2 = r_1/r_2e^{i(\alpha-\beta)}. \tag{8}$$

These show that when complex numbers are multiplied, their moduli are *multiplied* and their arguments are *added*, whereas when complex numbers are divided, their moduli are *divided* and their arguments are *subtracted*.

EXAMPLE 1.10

Find uv, u/v, and u^{25} given that $u = 1 + i$, $v = \sqrt{3} - i$.

Solution $u = 1 + i = \sqrt{2}e^{i\pi/4}$, $v = \sqrt{3} - i = 2e^{-i\pi/6}$, so $uv = 2\sqrt{2}e^{i\pi/12}$, $u/v = (1/\sqrt{2})e^{i5\pi/12}$ while $u^{25} = (\sqrt{2}e^{i\pi/4})^{25} = (\sqrt{2})^{25}(e^{i\pi/4})^{25} = 4096\sqrt{2}(e^{i(6+1/4)\pi}) = 4096\sqrt{2}(e^{i6\pi})(e^{i\pi/4}) = 4096\sqrt{2}(e^{i\pi/4}) = 4096\sqrt{2}(1 + i)$. ■

To find the principal value of the argument of a given complex number z, namely Arg z, use should be made of the signs of $x = \mathrm{Re}\{z\}$, and $y = \mathrm{Im}\{z\}$ together

with the results listed below, all of which follow by inspection of Fig. 1.5.

Signs of x and y	*Arg z = θ*
$x < 0,\ y < 0$	$-\pi < \theta < -\pi/2$
$x > 0,\ y < 0$	$-\pi/2 < \theta < 0$
$x > 0,\ y > 0$	$0 < \theta < \pi/2$
$x < 0,\ y > 0$	$\pi/2 < \theta < \pi$

EXAMPLE 1.11

Find $r = |z|$, Arg z, arg z, and the modulus–argument form of the following values of z.

(a) $-2\sqrt{3} - 2i$ (b) $-1 + i\sqrt{3}$ (c) $1 + i$ (d) $2 - i2\sqrt{3}$.

Solution **(a)** $r = \{(-2\sqrt{3})^2 + (-2)^2\}^{1/2} = 4$, Arg$z = \theta = -5\pi/6$ and arg $z = -5\pi/6 + 2k\pi$, $k = \pm 1, \pm 2, \ldots$, $z = 4(\cos(-5\pi/6) + i\sin(-5\pi/6))$.

(b) $r = \{(-1)^2 + (\sqrt{3})^2\}^{1/2} = 2$, Arg $z = \theta = 2\pi/3$ and arg $z = 2\pi/3 + 2k\pi$, $k = \pm 1, \pm 2, \ldots$, $z = 2(\cos(2\pi/3) + i\sin(2\pi/3))$.

(c) $r = \{(1)^2 + (1)^2\}^{1/2} = \sqrt{2}$, Arg $z = \theta = \pi/4$ and arg $z = \pi/4 + 2k\pi$, $k = \pm 1, \pm 2, \ldots$, $z = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$.

(d) $r = \{(2)^2 + (-2\sqrt{3})^2\}^{1/2} = 4$, Arg $z = \theta = -\pi/3$ and arg $z = -\pi/3 + 2k\pi$, $k = \pm 1, \pm 2, \ldots$, $z = 4(\cos(-\pi/3) + i\sin(-\pi/3))$. ■

EXERCISES 1.4

1. Expand $(\cos\theta + i\sin\theta)^2$ and then use trigonometric identities to show that

$$(\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta).$$

2. Give an inductive proof of de Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta),$$

for n a natural number.

3. Use de Moivre's theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

4. Use de Moivre's theorem to express $\cos 6\theta$ and $\sin 6\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

5. Show by expanding $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$ and using trigonometric identities that

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha + \beta) + i\sin(\alpha + \beta).$$

6. Show by expanding $(\cos\alpha + i\sin\alpha)/(\cos\beta + i\sin\beta)$ and using trigonometric identities that

$$(\cos\alpha + i\sin\alpha)/(\cos\beta + i\sin\beta) = \cos(\alpha - \beta) + i\sin(\alpha - \beta).$$

7. If $z = \cos\theta + i\sin\theta = e^{i\theta}$, show that when n is a natural number,

$$\cos(n\theta) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \quad \text{and} \quad \sin(n\theta) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right).$$

Use these results to express $\cos^3\theta\sin^3\theta$ in terms of multiple angles of θ. Hint: $\bar{z} = 1/z$.

8. Use the method of Exercise 7 to express $\sin^6\theta$ in terms of multiple angles of θ.

9. By expanding $(z + 1/z)^4$, grouping terms, and using the method of Exercise 7, show that

$$\cos^4\theta = (1/8)(3 + 4\cos 2\theta + \cos 4\theta).$$

10. By expanding $(z - 1/z)^5$, grouping terms, and using the method of Exercise 7, show that

$$\sin^5\theta = (1/16)(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

11. Use the method of Exercise 7 to show that

$$\cos^3\theta + \sin^3\theta = (1/4)(\cos 3\theta + 3\cos\theta - \sin^3\theta + 3\sin\theta).$$

In Exercises 12 through 15 express the functions of u, v, and w in modulus-argument form.

12. uv, u/v, and v^5, given that $u = 2 - 2i$ and $v = 3 + i3\sqrt{3}$.

13. uv, u/v, and u^7, given that $u = -1 - i\sqrt{3}$, $v = -4 + 4i$.

14. uv, u/v, and v^6, given that $u = 2 - 2i$, $v = 2 - i2\sqrt{3}$.

15. uvw, uw/v, and w^3/u^4, given that $u = 2 - 2i$, $v = 3 - i3\sqrt{3}$ and $w = 1 + i$.

16. Express $[(-8 + i8\sqrt{3})/(-1 - i)]^2$ in modulus–argument form.

17. Find in modulus–argument form $[(1 + i\sqrt{3})^3/(-1 + i)^2]^3$.

18. Use the factorization

$$(1 - z^{n+1}) = (1 - z)(1 + z + z^2 + \cdots + z^n) \qquad (z \neq 1)$$

with $z = e^{i\theta} = \exp(i\theta)$ to show that

$$\sum_{k=1}^{n} \exp(ik\theta) = \frac{\exp(in\theta) - 1}{1 - \exp(-i\theta)}.$$

19. Use the final result of Exercise 18 to show that

$$\sum_{k=1}^{n} \exp(ik\theta) = \frac{\exp[i(n + 1/2)\theta] - \exp(i\theta/2)}{\exp(i\theta/2) - \exp(-i\theta/2)},$$

and then use the result to deduce the **Lagrange identity**

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$$
$$= 1/2 + \frac{\sin[(n + 1/2)\theta]}{2\sin(\theta/2)}, \qquad \text{for } 0 < \theta < 2\pi.$$

1.5 Roots of Complex Numbers

It is often necessary to find the n values of $z^{1/n}$ when n is a positive integer and z is an arbitrary complex number. This process is called finding the ***n*th roots** of z. To determine these roots we start by setting

$$w = z^{1/n}, \quad \text{which is equivalent to } w^n = z.$$

Then, after defining w and z in modulus–argument form as

$$w = \rho e^{i\phi} \quad \text{and} \quad z = re^{i\theta}, \tag{9}$$

we substitute for w and z in $w^n = z$ to obtain

$$\rho^n e^{in\phi} = re^{i\theta}.$$

It is at this stage, in order to find all n roots, that use must be made of the many-valued nature of the argument of a complex number by recognizing that $1 = e^{2k\pi i}$ for $k = 0, \pm 1, \pm 2, \ldots$. Using this result we now multiply the right-hand side of the foregoing result by by $e^{2k\pi i}$ (that is, by 1) to obtain

$$\rho^n e^{in\phi} = re^{i\theta}e^{2k\pi i} = re^{i(\theta + 2k\pi)}.$$

Equality of complex numbers in modulus–argument form means the equality of their moduli and, correspondingly, the equality of their arguments, so applying this to the last result we have

$$\rho^n = r \quad \text{and} \quad n\phi = \theta + 2k\pi,$$

showing that

$$\rho = r^{1/n} \quad \text{and} \quad \phi = (\theta + 2k\pi)/n.$$

Here $r^{1/n}$ is simply the nth positive root of r: $\rho = \sqrt[n]{r}$.

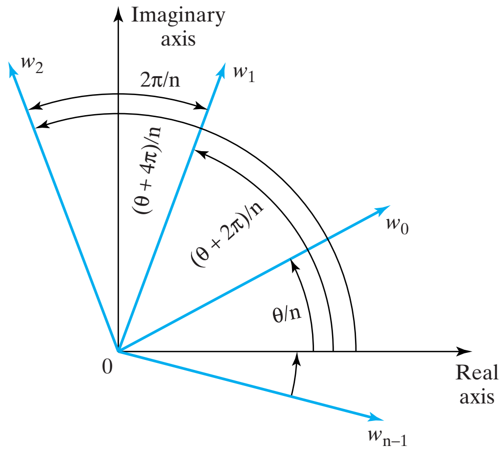


FIGURE 1.6 Location of the roots of $z^{1/n}$.

n*th roots of a complex number *z

Finally, when we substitute these results into the expression for w, we see that the n values of the roots denoted by $w_0, w_1, \ldots, w_{n-1}$ are given by

$$w_k = r^{1/n}\{\cos[(\theta + 2k\pi)/n] + i \sin[(\theta + 2k\pi)/n]\}, \quad \text{for } k = 0, 1, \ldots, n-1. \tag{10}$$

Notice that it is only necessary to allow k to run through the successive integers $0, 1, \ldots, n-1$, because the period of the sine and cosine functions is 2π, so allowing k to increase beyond the value $n-1$ will simply repeat this *same* set of roots. An identical argument shows that allowing k to run through successive negative integers can again only generate the same n roots $w_0, w_1, \ldots, w_{n-1}$.

Examination of the arguments of the roots shows them to be spaced uniformly around a circle of radius $r^{1/n}$ centered on the origin. The angle between the radial lines drawn from the origin to each successive root is $2\pi/n$, with the radial line from the origin to the first root w_0 making an angle θ/n to the positive real axis, as shown in Fig. 1.6. This means that if the location on the circle of any one root is known, then the locations of the others follow immediately.

Writing unity in the form $1 = e^{i0}$ shows its modulus to be $r = 1$ and the principal value of its argument to be $\theta = 0$. Substitution in formula (10) then shows the n roots of $1^{1/n}$, called the ***n*th roots of unity**, to be

$$w_0 = 1, \quad w_1 = e^{i\pi/n}, \quad w_2 = e^{i2\pi/n}, \ldots, w_{n-1} = e^{i(n-1)\pi/n}. \tag{11}$$

By way of example, the fifth roots of unity are located around the unit circle as shown in Fig. 1.7.

If we set $\omega = w_1$, it follows that the nth roots of unity can be written in the form

$$1, \omega, \omega^2, \ldots, \omega^{n-1}.$$

As $\omega^n = 1$ and $\omega^n - 1 = (\omega - 1)(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) = 0$, as $\omega_1 \neq 1$ we see that the the nth roots of unity satisfy

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0. \tag{12}$$

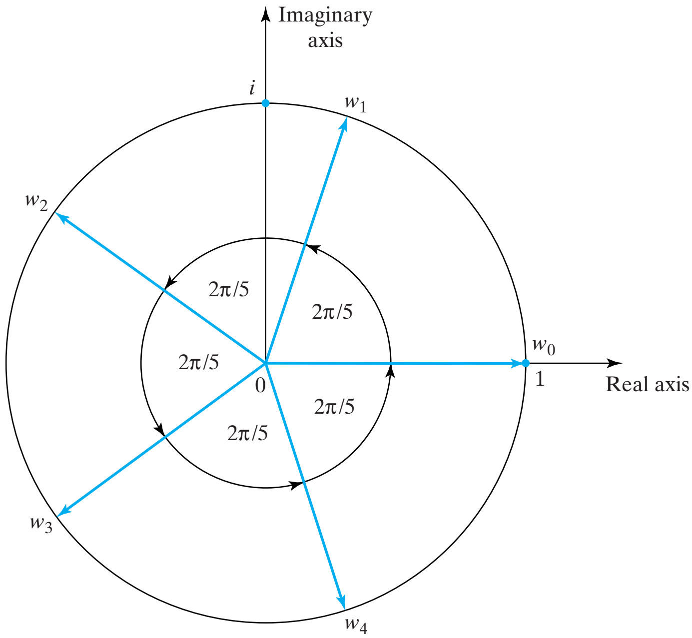


FIGURE 1.7 The fifth roots of unity.

This result remains true if ω is replaced by any one of the other nth roots of unity, with the exception of 1 itself.

EXAMPLE 1.12 Find $w = (1+i)^{1/3}$.

Solution Setting $z = 1 + i = \sqrt{2}e^{i\pi/4}$ shows that $r = |z| = \sqrt{2}$ and $\theta = \pi/4$. Substituting these results into formula (1) gives

$$w_k = 2^{1/6}\{\cos[(1/12)(1+8k)\pi] + i\sin[(1/12)(1+8k)\pi]\}, \quad \text{for } k = 0, 1, 2. \quad ■$$

The square root of a complex number $\zeta = \alpha + i\beta$ is often required, so we now derive a useful formula for its two roots in terms of $|\zeta|$, α and the sign of β. To obtain the result we consider the equation

$$z^2 = \zeta, \quad \text{where } \zeta = \alpha + i\beta,$$

and let Arg $\zeta = \theta$. Then we may write

$$z^2 = |\zeta|e^{i\theta},$$

and taking the square root of this result we find the two square roots z_- and z_+ are given by

$$\begin{aligned} z_\pm &= \pm|\zeta|^{1/2}e^{i\theta/2} \\ &= \pm|\zeta|^{1/2}\{\cos(\theta/2) + i\sin(\theta/2)\}. \end{aligned}$$

Now $\cos\theta = \alpha/|\zeta|$, but

$$\cos^2(\theta/2) = (1/2)(1+\cos\theta), \quad \text{and} \quad \sin^2(\theta/2) = (1/2)(1-\cos\theta),$$

so

$$\cos^2(\theta/2) = (1/2)(1 + \alpha/|\zeta|), \quad \text{and} \quad \sin^2(\theta/2) = (1/2)(1 - \alpha/|\zeta|).$$

As $-\pi < \theta \le \pi$, it follows that in this interval $\cos(\theta/2)$ is nonnegative, so taking the square root of $\cos^2(\theta/2)$ we obtain

$$\cos(\theta/2) = \left(\frac{|\zeta| + \alpha}{2|\zeta|}\right)^{1/2}.$$

However, the function $\sin(\theta/2)$ is negative in the interval $-\pi < \theta < 0$ and positive in the interval $0 < \theta < \pi$, and so has the same sign as β. Thus, the square root of $\sin^2(\theta/2)$ can be written in the form

$$\sin(\theta/2) = \operatorname{sign}(\beta)\left(\frac{|\zeta| - \alpha}{2|\zeta|}\right)^{1/2}.$$

Using these expressions for $\cos(\theta/2)$ and $\sin(\theta/2)$ in the square roots $z_{\pm}$ brings us to the following useful rule.

Rule for finding the square root of a complex number

Let $z^2 = \zeta$, with $\zeta = \alpha + i\beta$. Then the square roots z_+ and z_- of ζ are given by

$$z_+ = \left(\frac{|\zeta| + \alpha}{2}\right)^{1/2} + i\operatorname{sign}(\beta)\left(\frac{|\zeta| - \alpha}{2}\right)^{1/2}$$

$$z_- = -\left(\frac{|\zeta| + \alpha}{2}\right)^{1/2} - i\operatorname{sign}(\beta)\left(\frac{|\zeta| - \alpha}{2}\right)^{1/2}.$$

EXAMPLE 1.13 Find the square roots of (a) $\zeta = 1 + i$ and (b) $\zeta = 1 - i$.

Solution **(a)** $\zeta = 1 + i$ so $|\zeta| = \sqrt{2}$, $\alpha = 1$ and $\operatorname{sign}(\beta) = 1$, so the square roots of $\zeta = 1 + i$ are

$$z_{\pm} = \pm\left\{\left(\frac{\sqrt{2}+1}{2}\right)^{1/2} + i\left(\frac{\sqrt{2}-1}{2}\right)^{1/2}\right\}.$$

(b) $\zeta = 1 - i$, so $|\zeta| = \sqrt{2}$, $\alpha = 1$ and $\operatorname{sign}(\beta) = -1$, from which it follows that the square roots of $\zeta = 1 - i$ are

$$z_{\pm} = \pm\left\{\left(\frac{\sqrt{2}+1}{2}\right)^{1/2} - i\left(\frac{\sqrt{2}-1}{2}\right)^{1/2}\right\}.$$

■

The theorem that follows provides information about the roots of polynomials with *real* coefficients that proves to be useful in a variety of ways.

THEOREM 1.2 **Roots of a polynomial with real coefficients** Let

$$P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots a_{n-1} z + a_n$$

be a polynomial of **degree** n in which all the coefficients $a_1, a_2, \ldots, a_n$ are real. Then either all the n **roots** of $P(z) = 0$ are real, that is, the n **zeros** of $P(z)$ are all real, or any that are complex must occur in complex conjugate pairs.

Proof The proof uses the following simple properties of the complex conjugate operation.

1. If a is real, then $\overline{a} = a$. This result follows directly from the definition of the complex conjugate operation.
2. If b and c are any two complex numbers, then $\overline{b + c} = \overline{b} + \overline{c}$. This result also follows directly from the definition of the complex conjugate operation.
3. If b and c are any two complex numbers, then $\overline{bc} = \overline{b}\overline{c}$ and $\overline{b^r} = (\overline{b})^r$.

We now proceed to the proof. Taking the complex conjugate of $P(z) = 0$ gives

$$\overline{z}^n + \overline{a_1 z^{n-1}} + \overline{a_2 z^{n-2}} + \cdots + \overline{a_{n-1} z} + \overline{a_n} = 0,$$

but the a_r are all real so $\overline{a_r z^{n-r}} = \overline{a_r}\,\overline{z^{n-r}} = a_r \overline{z^{n-r}} = a_r (\overline{z})^{n-r}$, allowing the preceding equation to be rewritten as

$$(\overline{z})^n + a_1 (\overline{z})^{n-1} + a_2 (\overline{z})^{n-2} + \cdots + a_{n-1} \overline{z} + a_n = 0.$$

This result is simply $P(\overline{z}) = 0$, showing that if z is a complex root of $P(z)$, then so also is $\overline{z}$. Equivalently, z and $\overline{z}$ are both zeros of $P(z)$.

If, however, z is a real root, then $z = \overline{z}$ and the result remains true, so the first part of the theorem is proved. The second part follows from the fact that if $z = \alpha + i\beta$ is a root, then so also is $z = \alpha - i\beta$, and so $(z - \alpha - i\beta)$ and $(z - \alpha + i\beta)$ are factors of $P(z)$. The product of these factors must also be a factor of $P(z)$, but

$$(z - \alpha - i\beta)(z - \alpha + i\beta) = z^2 - 2\alpha z + \alpha^2 + \beta^2,$$

and the expression on the right is a quadratic in z with real coefficients, so the final result of the theorem is established. ■

EXAMPLE 1.14 Find the roots of $z^3 - z^2 - z - 2 = 0$, given that $z = 2$ is a root.

Solution If $z = 2$ is a root of $P(z) = 0$, then $z - 2$ is a factor of $P(z)$, so dividing $P(z)$ by $z - 2$ we obtain $z^2 + z + 1$. The remaining two roots of $P(z) = 0$ are the roots of $z^2 + z + 1 = 0$. Solving this quadratic equation we find that $z = (-1 \pm i\sqrt{3})/2$, so the three roots of the equation are 2, $(-1 + i\sqrt{3})/2$, and $(-1 - i\sqrt{3})/2$. ■

For more background information and examples on complex numbers, the complex plane and roots of complex numbers, see Chapter 1 of reference [6.1], Sections 1.1 to 1.5 of reference [6.4], and Chapter 1 of reference [6.6].

EXERCISES 1.5

In Exercises 1 through 8 find the square roots of the given complex number by using result (10), and then confirm the result by using the formula for finding the square root of a complex mumber.

1. $-1+i$.
2. $3+2i$.
3. i.
4. $-1+4i$.
5. $2-3i$.
6. $-2-i$.
7. $4-3i$.
8. $-5+i$.

In Exercises 9 through 14 find the roots of the given complex number.

9. $(1+i\sqrt{3})^{1/3}$.
10. $i^{1/4}$.
11. $(-1)^{1/4}$.
12. $(-1-i)^{1/3}$.
13. $(-i)^{1/3}$.
14. $(4+4i)^{1/4}$.
15. Find the roots of $z^3+z(i-1)=0$.
16. Find the roots of $z^3+iz/(1+i)=0$.
17. Use result (12) to show that
$$1+\cos(2\pi/n)+\cos(4\pi/n)+\cdots$$
$$+\cos[(2(n-1)\pi/n)]=0$$
and
$$\sin(2\pi/n)+\sin(4\pi/n)+\cdots+\sin[(2(n-1)\pi/n)]=0.$$
18. Use Theorem 1.1 and the representation $z=re^{i\theta}$ to prove that if a and b are any two arbitrary complex numbers, then $\overline{ab}=\overline{a}\overline{b}$ and $\overline{(a^r)}=(\overline{a})^r$.
19. Given $z=1$ is a zero of the polynomial $P(z)=z^3-5z^2+17z-13$, find its other two zeros and verify that they are complex conjugates.
20. Given that $z=-2$ is a zero of the polynomial $P(z)=z^5+2z^4-4z-8$, find its other four zeros and verify that they occur in complex conjugate pairs.
21. Find the two zeros of the quadratic $P(z)=z^2-1+i$, and explain why they do not occur as a complex conjugate pair.

1.6 Partial Fractions

Let $N(x)$ and $D(x)$ be two polynomials. Then a **rational function** of x is any function of the form $N(x)/D(x)$. The method of **partial fractions** involves the decomposition of rational functions into an equivalent sum of simpler terms of the type

$$\frac{P_1}{ax+b},\ \frac{P_2}{(ax+b)^2},\ldots \quad \text{and} \quad \frac{Q_1x+R_1}{Ax^2+Bx+C},\ \frac{Q_2x+R_2}{(Ax^2+Bx+C)^2},\ldots,$$

where the coefficients are all real together with, possibly, a polynomial in x.

The steps in the reduction of a rational function to its partial fraction representation are as follows:

STEP 1 Factorize $D(x)$ into a product of linear factors and quadratic factors with real coefficients with complex roots, called **irreducible** factors. This is the hardest step, and real quadratic factors will only arise when $D(x)=0$ has pairs of complex conjugate roots (see Theorem 1.2). Use the result to express $D(x)$ in the form

$$D(x)=(a_1x+b_1)^{r_1}\ldots(a_mx+b_m)^{r_m}\left(A_1x^2+B_1x+C_1\right)^{s_1}$$
$$\ldots\left(A_kx^2+B_kx+C_k\right)^{s_k},$$

where r_i is the number of times the linear factor (a_ix+b_i) occurs in the factorization of $D(x)$, called its **multiplicity**, and s_j is the corresponding multiplicity of the quadratic factor $(A_jx^2+B_jx+C_j)$.

STEP 2 Suppose first that the degree n of the numerator is *less* than the degree d of the denominator. Then, to every different linear factor $(ax + b)$ with multiplicity r, include in the partial fraction expansion the terms

$$\frac{P_1}{(ax+b)} + \frac{P_2}{(ax+b)^2} + \cdots + \frac{P_r}{(ax+b)^r},$$

where the constant coefficients P_i are unknown at this stage, and so are called **undetermined coefficients**.

partial fraction undetermined coefficients

STEP 3 To every quadratic factor $(Ax^2 + Bx + C)^s$ with multiplicity s include in the partial fraction expansion the terms

$$\frac{Q_1x + R_1}{(Ax^2+Bx+C)} + \frac{Q_2x + R_2}{(Ax^2+Bx+C)^2} + \cdots + \frac{Q_sx + R_s}{(Ax^2+Bx+C)^s},$$

where the Q_j and R_j for $j = 1, 2, \ldots, s$ are undetermined coefficients.

STEP 4 Take as the partial fraction representation of $N(x)/D(x)$ the sum of all the terms in Steps 2 and 3.

STEP 5 Multiply the expression

$$N(x)/D(x) = \text{Partial fraction representation in Step 4}$$

by $D(x)$, and determine the unknown coefficients by equating the coefficients of corresponding powers of x on either side of this expression to make it an identity (that is, true for all x).

STEP 6 Substitute the values of the coefficients determined in Step 5 into the expression in Step 4 to obtain the required partial fraction representation.

STEP 7 If $n \geq d$, use long division to divide the denominator into the numerator to obtain the sum of a polynomial of degree $n - d$ of the form

$$T_0 + T_1x + T_2x^2 + \cdots + T_{n-d}x^{n-d},$$

together with a remainder term in the form of a rational function $R(x)$ of the type just considered. Find the partial fraction representation of the rational function $R(x)$ using Steps 1 to 6. The required partial fraction representation is then the sum of the polynomial found by long division and the partial fraction representation of $R(x)$.

EXAMPLE 1.15 Find the partial fraction representations of

$$\text{(a) } F(x) = \frac{x^2}{(x+1)(x-2)(x+3)} \quad \text{and} \quad \text{(b) } F(x) = \frac{2x^3 - 4x^2 + 3x + 1}{(x-1)^2}.$$

Solution **(a)** All terms in the denominator are linear factors, so by Step 1 the appropriate form of partial fraction representation is

$$\frac{x^2}{(x+1)(x-2)(x+3)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+3}.$$

Cross multiplying, we obtain

$$x^2 = A(x-2)(x+3) + B(x+1)(x+3) + C(x+1)(x-2).$$

Setting $x = -1$ makes the terms in B and C vanish and gives $A = -1/6$. Setting $x = 2$ makes the terms in A and C vanish and gives $B = 4/15$, whereas setting $x = -3$ makes the terms in A and B vanish and gives $C = 9/10$, so

$$\frac{x^2}{(x+1)(x-2)(x+3)} = \frac{-1}{6(x+1)} + \frac{4}{15(x-2)} + \frac{9}{10(x+3)}.$$

(b) The degree of the numerator exceeds that of the denominator, so from Step 7 it is necessary to start by dividing the denominator into the numerator longhand to obtain

$$\frac{2x^3 - 4x^2 + x + 3}{(x-1)^2} = 2x + \frac{3-x}{(x-1)^2}.$$

We now seek a partial fraction representation of $(3-x)/(x-1)^2$ by using Step 1 and writing

$$\frac{3-x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

When we multiply by $(x-1)^2$, this becomes

$$3 - x = A(x-1) + B.$$

Equating the constant terms gives $3 = -A + B$, whereas equating the coefficients of x gives $-1 = A$ so that $B = 2$. Thus, the required partial fraction representation is

$$\frac{2x^3 - 4x^2 + x + 3}{(x-1)^2} = 2x + \frac{1}{1-x} + \frac{2}{(x-1)^2}.$$ ■

An examination of the way the undetermined coefficients were obtained in (a) earlier, where the degree of the numerator is less than that of the denominator and linear factors occur in the denominator, leads to a simple rule for finding the undetermined coefficients called the "cover-up rule."

The cover-up rule

Let a partial fraction decomposition be required for a rational function $N(x)/D(x)$ in which the degree of the numerator $N(x)$ is less than that of the denominator $D(x)$ and, when factored, let $D(x)$ contain some linear factors (factors of degree 1).

Let $(x-\alpha)$ be a linear factor of $D(x)$. Then the unknown coefficient K in the term $K/(x-\alpha)$ in the partial fraction decomposition of $N(x)/D(x)$ is obtained by "covering up" (ignoring) all of the other terms in the partial fraction expansion, multiplying the remaining expression $N(x)/D(x) = K/(x-\alpha)$ by $(x-\alpha)$, and then determining K by setting $x = \alpha$ in the result.

To illustrate the use of this rule we use it in case (a) given earlier to find A from the representation

$$\frac{x^2}{(x+1)(x-2)(x+3)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+3}.$$

We "cover up" (ignore) the terms involving B and C, multiply through by $(x+1)$, and find A from the result

$$\frac{x^2}{(x-2)(x+3)} = A$$

by setting $x = -1$, when we obtain $A = -1/6$. The undetermined coefficients B and C follow in similar fashion.

completing the square

Once a partial fraction representation of a function has been obtained, it is often necessary to express any quadratic $x^2 + px + q$ that occurs in a denominator in the form $(x + A)^2 + B$, where A and B may be either positive or negative real numbers. This is called **completing the square**, and it is used, for example, when integrating rational functions and when finding inverse Laplace transforms.

To find A and B we set

$$\begin{aligned} x^2 + px + q &= (x + A)^2 + B \\ &= x^2 + 2Ax + A^2 + B, \end{aligned}$$

and to make this an identity we now equate the coefficients of corresponding powers of x on either side of this expression:

$$\begin{array}{ll} (\text{coefficients of } x^2) & 1 = 1 \text{ (this tells us nothing)} \\ (\text{coefficients of } x) & p = 2A \\ (\text{constant terms}) & q = A^2 + B. \end{array}$$

Consequently $A = (1/2)p$ and $B = q - (1/4)p^2$, and so the result obtained by completing the square is

$$x^2 + px + q = [x + (1/2)p]^2 + q - (1/4)p^2.$$

If the more general quadratic $ax^2 + bx + c$ occurs, all that is necessary to reduce it to this same form is to write it as

$$ax^2 + bx + c = a[x^2 + (b/a)x + c/a],$$

and then to complete the square using $p = b/a$ and $q = c/a$.

EXAMPLE 1.16

Complete the square in the following expressions:

(a) $x^2 + x + 1$.
(b) $x^2 + 4x$.
(c) $3x^2 + 2x + 1$.

Solution **(a)** $p = 1, q = 1$, so $A = 1/2$, $B = 3/4$, and hence

$$x^2 + x + 1 = (x + 1/2)^2 + 3/4.$$

(b) $p = 4,\ q = 0$, so $A = 2$, $B = -4$, and hence

$$x^2 + 4x = (x + 2)^2 - 4.$$

(c) $3x^2 + 2x + 1 = 3[x^2 + (2/3)x + 1/3]$ and so $p = 2/3$, $q = 1/3$, from which it follows that $A = 1/3$ and $B = 2/9$, so

$$3x^2 + 2x + 1 = 3\{(x + 1/3)^2 + 2/9\}.$$

Further information and examples of partial fractions can be found in any one of references [1.1] to [1.7]. ■

EXERCISES 1.6

Express the rational functions in Exercises 1 through 8 in terms of partial fractions using the method of Section 1.6, and verify the results by using computer algebra to determine the partial fractions.

1. $(3x+4)/(2x^2+5x+2)$.
2. $(x^2+3x+5)/(2x^2+5x+3)$.
3. $(3x-7)/(2x^2+9x+10)$.
4. $(x^2+3x+2)/(x^2+2x-3)$.
5. $(x^3+x^2+x+1)/[(x+2)^2(x^2+1)]$.
6. $(x^2-1)/(x^2+x+1)$.
7. $(x^3+x^2+x+1)/\{(x+2)^2(x+1)\}$.
8. $(x^2+4)/(x^3+3x^2+3x+1)$.

Complete the square in Exercises 9 through 14.

9. x^2+4x+5.
10. x^2+6x+7.
11. $2x^2+3x-6$.
12. $4x^2-4x-3$.
13. $2-2x+9x^2$.
14. $2+2x-x^2$.

1.7 Fundamentals of Determinants

A **determinant** of **order** $\boldsymbol{n}$ is a single number associated with an array $\mathbf{A}$ of n^2 numbers arranged in n rows and n columns. If the number in the ith row and jth column of a determinant is a_{ij}, the determinant of $\mathbf{A}$, denoted by det $\mathbf{A}$ and sometimes by $|\mathbf{A}|$, is written

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}. \tag{13}$$

It is customary to refer to the entries a_{ij} in a determinant as its *elements*. Notice the use of vertical bars enclosing the array $\mathbf{A}$ in the notation $|\mathbf{A}|$ for the *determinant* of $\mathbf{A}$, as opposed to the use of the square brackets in $[\mathbf{A}]$ that will be used later to denote the *matrix* associated with an array $\mathbf{A}$ of quantities in which the number of rows need not be equal to the number of columns.

The value of a first order determinant det $\mathbf{A}$ with the single element a_{11} is defined as a_{11} so that $\det[a_{11}] = a_{11}$ or, in terms of the alternative notation for a determinant, $|a_{11}| = a_{11}$. This use of the notation $|.|$ to signify a determinant should not be confused with the notation used to signify the absolute value of a number.

The second order determinant associated with an array of elements containing two rows and two columns is defined as

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}, \tag{14}$$

so, for example, using the alternative notation for a determinant we have

$$\begin{vmatrix} 9 & 3 \\ -7 & -4 \end{vmatrix} = 9(-4) - (-7)3 = -15.$$

Notice that **interchanging** two rows or columns of a determinant changes its sign.

We now introduce the terms *minor* and *cofactor* that are used in connection with determinants of all orders, and to do so we consider the third order determinant

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{15}$$

minors and cofactors

The **minor** M_{ij} associated with a_{ij}, the element in the ith row and jth column of det $\mathbf{A}$, is defined as the second order determinant obtained from det $\mathbf{A}$ by deleting the elements (numbers) in its ith row and jth column. The **cofactor** C_{ij} of an element in the ith row and jth column of the det $\mathbf{A}$ in (15) is defined as the **signed minor** using the rule

$$C_{ij} = (-1)^{i+j} M_{ij}. \tag{16}$$

With these ideas in mind, the determinant det $\mathbf{A}$ in (15) is defined as

$$\begin{aligned}\det \mathbf{A} &= \sum_{j=1}^{3} a_{1j}(-1)^{1+j} \det M_{1j} \\ &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}.\end{aligned}$$

If we introduce the cofactors C_{ij}, this last result can be written

$$\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}, \tag{17}$$

and more concisely as

$$\det \mathbf{A} = \sum_{j=1}^{3} a_{1j}C_{1j}. \tag{18}$$

Result (18), or equivalently (17), will be taken as the definition of a third order determinant.

EXAMPLE 1.17 Evaluate the determinant

$$\begin{vmatrix} 1 & 3 & -3 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{vmatrix}.$$

Solution

The minor $M_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = (1)(1) - (0)(1) = 1$, so the cofactor $C_{11} = (-1)^{(1+1)}M_{11} = 1$.
The minor $M_{12} = \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = (2)(1) - (0)(-2) = 2$, so the cofactor $C_{12} = (-1)^{(1+2)}M_{12} = -2$.
The minor $M_{13} = \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} = (2)(1) - (1)(-2) = 4$, so the cofactor $C_{13} = (-1)^{(1+3)}M_{13} = 4$.

Using (17) we have

$$\begin{vmatrix} 1 & 3 & -3 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{vmatrix} = (1)C_{11} + (3)C_{12} + (-3)C_{13} = (1)(1) + (3)(-2) + (-3)(4) = -17.$$

■

When expanded, (17) becomes

$$\det \mathbf{A} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22},$$

and after regrouping these terms in the form

$$\det \mathbf{A} = -a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{22}a_{11}a_{33} - a_{22}a_{31}a_{13} - a_{23}a_{11}a_{32} + a_{23}a_{31}a_{12},$$

we find that

$$\det \mathbf{A} = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}.$$

Proceeding in this manner, we can easily show that det **A** may be obtained by forming the sum of the products of the elements of **A** and their cofactors in *any* row or column of det **A**. These results can be expressed symbolically as follows.

Expanding in terms of the elements of the *i*th row:

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} = \sum_{j=1}^{3} a_{ij}C_{ij}. \tag{19}$$

Laplace expansion theorem

Expanding in terms of the elements of the *j*th column:

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} = \sum_{i=1}^{3} a_{ij}C_{ij}. \tag{20}$$

Results (19) and (20) are the form taken by the **Laplace expansion theorem** when applied to a third order determinant. The extension of the theorem to determinants of any order will be made later in Chapter 3, Section 3.3.

EXAMPLE 1.18

Expand the following determinant (a) in terms of elements of its first row, and (b) in terms of elements of its third column:

$$|A| = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix}.$$

Solution **(a)** Expanding in terms of the elements of the first row requires the three cofactors $C_{11} = M_{11}$, $C_{12} = -M_{12}$, and $C_{13} = M_{13}$, where

$$M_{11} = \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = -4, \quad M_{12} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1, \quad M_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2,$$

so $C_{11} = (-1)^{(1+1)}(-4) = -4$, $C_{12} = (-1)^{(1+2)}(-1) = 1$, $C_{13} = (-1)^{(1+3)}(2) = 2$, and so

$$|A| = (1)(-4) + (2)(1) + (4)(2) = 6.$$

(b) Expanding in terms of the elements of the third column requires the three cofactors $C_{13} = M_{13}$, $C_{23} = -M_{23}$, and $C_{33} = M_{33}$, where

$$M_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2, \quad M_{23} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0, \quad M_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2,$$

so $C_{13} = (-1)^{(1+3)}(2) = 2$, $C_{23} = 0$, $C_{33} = (-1)^{(3+3)}(-2) = -2$ and so

$$|A| = (4)(2) + (2)(0) + (1)(-2) = 6.$$

■

Two especially simple *third order* determinants are of the form

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} \quad \text{and} \quad \det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

The first of these determinants has only zero elements below the diagonal line drawn from its top left element to its bottom right one, and the second determinant has only zero elements above this line. This diagonal line in every determinant is called the **leading diagonal**. The value of each of the preceding determinants is easily seen to be given by the product $a_{11}a_{22}a_{33}$ of the terms on its leading diagonal.

Simpler still in form is the third order determinant

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33},$$

whose value $a_{11}a_{22}a_{33}$ is again the product of the elements on the leading diagonal.

For another approach to the elementary properties of determinants, see Appendix A16 of reference [1.2], and Chapter 2 of reference [2.1].

EXERCISES 1.7

Evaluate the determinants in Exercises 1 through 6 (a) in terms of elements of the first row and (b) in terms of elements of the second column.

1. $\begin{vmatrix} 1 & 5 & 7 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$.

2. $\begin{vmatrix} 2 & 1 & -1 \\ 2 & 6 & -1 \\ 5 & 1 & -1 \end{vmatrix}$.

3. $\begin{vmatrix} 5 & 2 & 4 \\ 1 & 2 & 1 \\ 3 & 1 & 5 \end{vmatrix}$.

4. $\begin{vmatrix} -1 & 3 & 6 \\ 2 & 1 & 4 \\ -1 & 3 & 1 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 0 & -6 \\ 2 & 1 & 3 \\ 4 & 3 & 21 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & 5 & -1 \\ 2 & 1 & -3 \\ -4 & 1 & 1 \end{vmatrix}$.

7. On occasion the elements of a matrix may be functions, in which case the determinant may be a function. Evaluate the *functional* determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \sin x & -\cos x \\ 0 & \cos x & \sin x \end{vmatrix}.$$

8. Determine the values of λ that make the following determinant vanish:

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 2-\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix}.$$

Hint: This is a polynomial in λ of degree 3.

9. A matrix is said to be **transposed** if its first row is written as its first column, its second row is written as its second column . . . , and its last row is written as its last column. If the determinant is $|\mathbf{A}|$, the determinant of $\mathbf{A}^{\mathrm{T}}$, the transpose matrix $\mathbf{A}$, is denoted by $|\mathbf{A}^{\mathrm{T}}|$. Write out the expansion of $|\mathbf{A}|$ using (17) and reorder the terms to show that

$$|\mathbf{A}| = |\mathbf{A}^{\mathrm{T}}|.$$

10. Use elimination to solve the system of linear equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

for x_1 and x_2, in which not both b_1 and b_2 are zero, and show that the solution can be written in the form

$$x_1 = D_1/|\mathbf{A}| \quad \text{and} \quad x_2 = D_2/|\mathbf{A}|, \quad \text{provided } |\mathbf{A}| \neq 0,$$

where $|\mathbf{A}|$ is the determinant of the matrix of *coefficients* of the system

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad \text{and} \quad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

Notice that D_1 is obtained from $|\mathbf{A}|$ by replacing its *first* column by b_1 and b_2, whereas D_2 is obtained from $|\mathbf{A}|$ by replacing its *second* column by b_1 and b_2. This is **Cramer's rule** for a system of two simultaneous equations. Use this method to find the solution of

$$x_1 + 5x_2 = 3$$
$$7x_1 - 3x_2 = -1.$$

11. Repeat the calculation in Exercise 10 using the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{aligned}$$

in which not all of b_1, b_2, and b_3 are zero, and show that provided $|\mathbf{A}| \neq 0$,

$$x_1 = D_1/|\mathbf{A}|, \quad x_2 = D_2/|\mathbf{A}|, \quad \text{and} \quad x_3 = D_3/|\mathbf{A}|,$$

where $|\mathbf{A}|$ is the determinant of the matrix of coefficients and D_i is the determinant obtained from $|\mathbf{A}|$ by replacing its ith column by b_1, b_2, and b_3 for $i = 1, 2, 3$. This is **Cramer's rule** for a system of three simultaneous equations, and the method generalizes to a system of n linear equations in n unknowns. Use this method to find the solution of

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_1 - 3x_2 - 2x_3 &= -1 \\ 2x_1 + x_2 + 2x_3 &= 1. \end{aligned}$$

1.8 Continuity in One or More Variables

If the function $y = f(x)$ is defined in the interval $a \leq x \leq b$, the interval is called the **domain of definition** of the function. The function f is said to have a **limit** at a point c in $a \leq x \leq b$, written $\lim_{x \to c} f(x) = L$, if for every arbitrarily small number $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{when } |x - c| < \delta. \tag{21}$$

This technical definition means that as x either increases toward c and becomes arbitrarily close to it, or decreases toward c and becomes arbitrarily close to it, so $f(x)$ approaches arbitrarily close to the value L. Notice that it is not necessary for $f(x)$ to be defined at $x = c$, or, if it is, that $f(c)$ assumes the value L. If $f(x)$ has a limit L as $x \to c$ and in addition $f(c) = L$, so that

$$\lim_{x \to c} f(x) = f(c) = L, \tag{22}$$

then the function f is said to be **continuous** at c. It must be emphasized that in this definition of continuity the limiting operation $x \to c$ must be true as x tends to c from *both* the left and right. It is convenient to say that x approaches c from the *left* when it increases toward c and, correspondingly, to say that x approaches c from the *right* when it decreases toward it.

continuity from the right

The function f is **continuous from the right** at $x = c$ if

$$\lim_{x \to c^+} f(x) = f(c), \tag{23}$$

where the notation $x \to c^+$ means that x decreases toward c, causing x to tend to c from the *right*. Similarly, f is **continuous from the left** at $x = c$ if

continuity from the left

$$\lim_{x \to c^-} f(x) = f(c), \tag{24}$$

where now $x \to c^-$ means that x increases toward c, causing x to tend to c from the *left*. The relationship among definitions (22), (23), and (24) is that f is continuous at the point c if

continuity at $x = c$

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c). \tag{25}$$

When expressed in words, this says that f is continuous at $x = c$ if the limits of f as x tends to c from both the left and right exist and, furthermore, the limits equal the functional value $f(c)$.

continuous function

A function f that is continuous at all points of $a \leq x \leq b$ is said to be a **continuous function** on that interval. Graphically, a continuous function on $a \leq x \leq b$ is a function whose graph is unbroken but not necessarily smooth. A function f is said

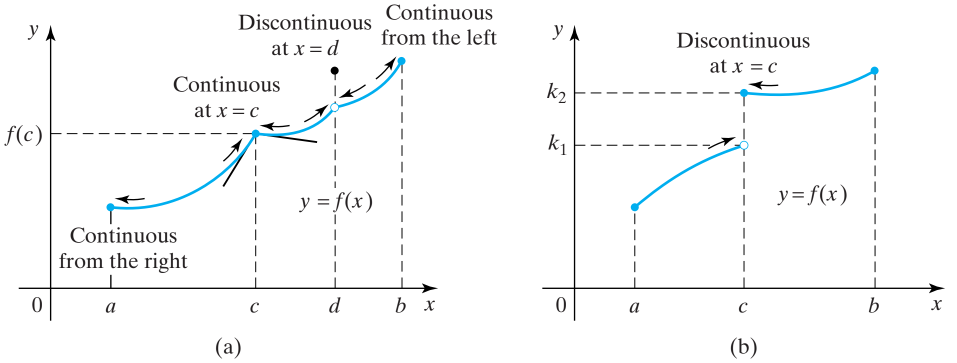


FIGURE 1.8 (a) A continuous function for $a < x < b$. (b) A discontinuous function.

smooth function

continuous and piecewise smooth function

discontinuous function

to be **smooth** over an interval if at each point of the graph the tangent lines to the left and right of the point are the same. Figure 1.8a shows the graph of a continuous function that is smooth over the intervals $a \le x < c$ and $c < x < b$ but has different tangent lines to the immediate left and right of $x = c$ where the function is *not* smooth. A function such as this is said to be **continuous and piecewise smooth** over the interval $a \le x \le b$.

A function f is said to be **discontinuous** at a point c if it is not continuous there. For a jump discontinuity we have

$$\lim_{x\to c^-} f(x) = k_1 \quad \text{and} \quad \lim_{x\to c^+} f(x) = k_2, \quad \text{but } k_1 \ne k_2. \tag{26}$$

A function f is said to have a **removable discontinuity** at a point c if $k_1 = k_2$ in (26), but $f(c) \ne k_1$, as at the point c_2 in Fig. 1.9.

An example of a discontinuous function is shown in Fig. 1.8b where a jump discontinuity occurs at $x = c$.

piecewise continuity

A function f is said to be **piecewise continuous** on an interval $a \le x \le b$ if it is continuous on a finite number of adjacent subintervals, but discontinuous at the end points of the subintervals, as shown in Fig. 1.9.

The notion of continuity of a function of several variables is best illustrated by considering a function $f(x, y)$ of the two independent variables x and y. The function f defined in some region of the (x, y)-plane D, say, is said to be **continuous**

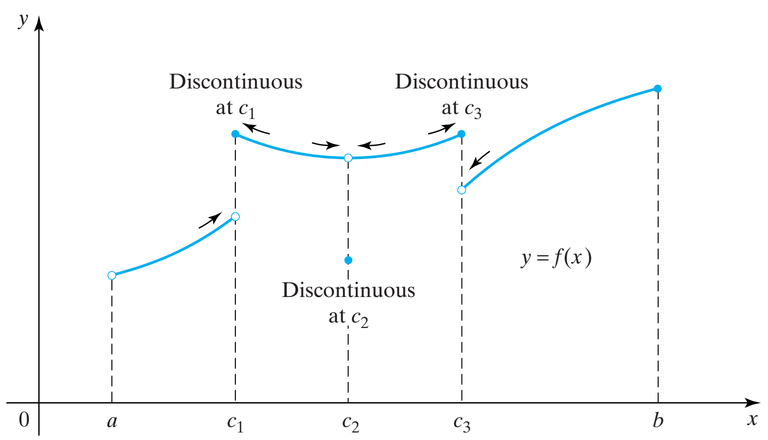


FIGURE 1.9 A piecewise continuous function.

at the point (a, b) in D if

continuity of $f(x, y)$

$$\lim_{x \to a, y \to b} f(x, y) = f(a, b), \tag{27}$$

and to be discontinuous otherwise.

In this definition of continuity, it is important to recognize that a general point P at (x, y) is allowed to tend to the point (a, b) in D along *any* path in the (x, y)-plane that lies in D. Expressed differently, f will only be continuous at (a, b) if the limit in (27) is independent of the way in which the point (x, y) approaches the point (a, b). When this is true for all points in D, the function f is said to be **continuous** in D.

discontinuity of $f(x, y)$

The function f is, for instance, **discontinuous** at (a, b) if

$$\lim_{x \to a, y \to b} f(x, y) = k, \quad \text{but } f(a, b) \neq k.$$

Sufficient for showing that a function f is discontinuous at a point (a, b) is by demonstrating that two *different* limiting values of f are obtained if the point P at (x, y) is allowed to tend to (a, b) along two *different* straight-line paths. This approach can be used to show that the function

$$f(x, y) = \frac{xy}{x^2 + a^2 y^2}$$

has no limit at the origin. If we allow the point P at (x, y) to tend to the origin along the straight line $y = kx$, with k an arbitrary constant, the function f becomes

$$f(x, kx) = \frac{k}{1 + a^2 k^2},$$

and it is seen from this that f is constant along each such line. However, the value of f on each line, and hence at the origin, depends on k, so f has no limit at the origin and so is discontinuous at that point, though f is defined and continuous at all other points of the (x, y)-plane.

An example of a function $f(x, y)$ that is continuous everywhere except at points along a curve Γ in the (x, y)-plane is shown in Fig. 1.10.

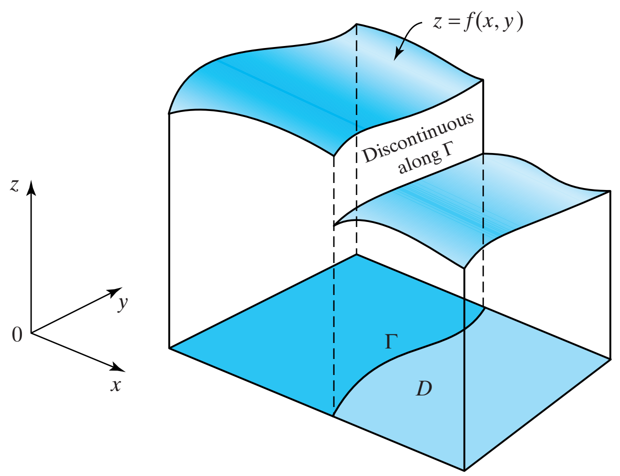


FIGURE 1.10 A function $f(x, y)$ continuous everywhere except at points on Γ.

The extension of these definitions to functions of n variables is immediate and so will not be discussed.

Discussions on continuity and its consequences can be found in any one of references [1.1] to [1.7].

1.9 Differentiability of Functions of One or More Variables

The function $f(x)$ defined in $a \le x \le b$ is said to be **differentiable** with the **derivative** $f'(c)$ at a point c inside the interval if the following limit exists:

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c). \tag{28}$$

differentiability of $f(x)$

Here, as in the definition of continuity, for f to be differentiable at point c the limit must remain unchanged as h tends to zero through both positive and negative values. The function f is said to be **differentiable** in the interval $a \le x \le b$ if it is differentiable at every point in the interval. When f is differentiable at a point c with derivative $f'(c)$, the number $f'(c)$ is the gradient, or slope, of the tangent line to the graph at the point $(c, f(c))$. A function with a continuous derivative throughout an interval is said to be a **smooth** function over the interval. The function f will be said to be **nondifferentiable** at any point c where the limit in (28) does not exist.

left- and right-hand derivatives of $f(x)$

Even when a function f is nondifferentiable at a point, it is possible that a special form of derivative can still be defined to the left and right of the point if the requirement that the limit in (28) exists as $h \to 0$ through both positive and negative values is relaxed. The function f has a **right-hand derivative** at a if the limit

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \tag{29}$$

exists, and a **left-hand derivative** at b if the limit

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}. \tag{30}$$

exists.

When c is a specific point, $f'(c)$ is a *number*, but when x is a variable, $f'(x)$ becomes a function. Left- and right-hand derivatives are illustrated in Fig. 1.11. An important consequence of differentiability is that **differentiability implies continuity**, but the converse is not true.

first order partial derivatives of $f(x, y)$

The **first order partial derivative with respect to x** of the function $f(x, y)$ of the two independent variables x and y at the point (a, b) is the number defined by

$$\lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}, \tag{31}$$

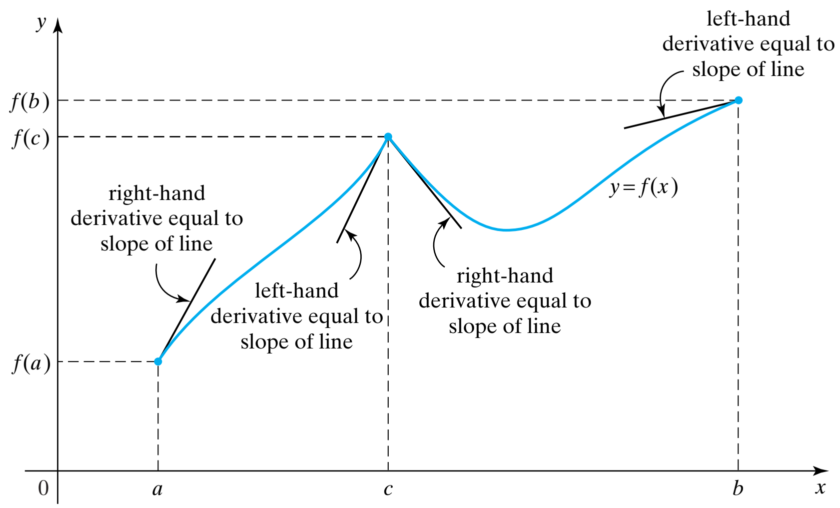


FIGURE 1.11 Left- and right-hand derivatives as tangent lines.

provided the limit exists. The value of this partial derivative is denoted either by $\partial f/\partial x$ at (a, b), or by $f_x(a, b)$. The corresponding partial derivative at a general point (x, y) is the function $f_x(x, y)$.

Similarly, the **first order partial derivative with respect to *y*** of the function $f(x, y)$ at the point (a, b) is the number defined by the limit

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a, b)}{k}, \tag{32}$$

provided the limit exists. The value of this partial derivative is denoted either by $\partial f/\partial y$ at (a, b), or by $f_y(a, b)$. At a general point (x, y) this partial derivative becomes the function $f_y(x, y)$. Higher order partial derivatives are defined in a similar fashion leading, for example, to the **second order partial derivatives**

second order partial derivatives of *f*(*x*, *y*)

$$\partial^2 f/\partial x^2 = \partial/\partial x(\partial f/\partial x),\ \partial^2 f/\partial y^2 = \partial/\partial y(\partial f/\partial y),$$
$$\partial^2 f/\partial x\partial y = \partial/\partial y(\partial f/\partial x), \quad \text{and} \quad \partial^2 f/\partial y\partial x = \partial/\partial x(\partial f/\partial y).$$

A more compact notation for these same derivatives is

$$f_{xx},\ f_{yy},\ f_{xy},\ \text{and}\ f_{yx},\ \text{so that, for example}\ f_{yx} = \partial^2 f/\partial y\partial x\ \text{and}\ f_{yy} = \partial^2 f/\partial y^2.$$

mixed partial derivatives

The derivatives f_{xy} and f_{yx} are called **mixed partial derivatives**, and their relationship forms the statement of the next theorem, the proof of which can be found in any one of references [1.1] to [1.7].

THEOREM 1.3

Equality of mixed partial derivatives Let f, f_x, f_{xy}, and f_{yx} all be defined and continuous at a point (a, b) in a region. Then

$$f_{xy}(a, b) = f_{yx}(a, b). \qquad \blacksquare$$

This result, given conditions for the *equality* of mixed partial derivatives, is an important one, and use will be made of it on numerous occasions as, for example, in Chapter 18 when second order partial differential equations are considered.

total differential

If $z = f(x, y)$, the **total differential** dz of f is defined as

$$dz = (\partial f/\partial x)\,dx + (\partial f/\partial y)\,dy, \tag{33}$$

where dz, dx, and dy are *differentials*. Here, a **differential** means a small quantity, and the differential dz is determined by (33) when the differentials dx and dy are specified. When $\partial f/\partial x$ and $\partial f/\partial y$ are evaluated at a specific point (a, b), result (33) provides a linear approximation to $f(x, y)$ near to the point (a, b). Although finite, the limits of the quotients of the differentials $dz \div dx$ and $dy \div dx$ as the differential $dx \to 0$ are such that they become the values of the derivatives dz/dx and dy/dx, respectively, at a point (x, y) where $\partial f/\partial x$ and $\partial f/\partial y$ are evaluated.

1.10 Tangent Line and Tangent Plane Approximations to Functions

tangent line approximation

Let $y = f(x)$ be defined in the interval $a \le x \le b$ and be differentiable throughout it. Then a **tangent line (linear) approximation** to f near a point x_0 in the interval is given by

$$y_{\mathrm{T}} = f(x_0) + (x - x_0)f'(x_0). \tag{34}$$

This linear expression approximates the function f close to x_0 by the tangent to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$.

This simple approximation has many uses; one will be in the Euler and modified Euler methods for solving initial value problems for ordinary differential equations developed in Chapter 19.

EXAMPLE 1.19

Find a tangent line approximation to $y = 1 + x^2 + \sin x$ near the point $x = \alpha$.

Solution Setting $x_0 = \alpha$ and substituting into (34) gives

$$y \approx 1 + \alpha^2 + \sin\alpha + (x - \alpha)(2\alpha + \cos\alpha) \text{ for } x \text{ close to } \alpha.$$

■

Let the function $z = f(x, y)$ be defined in a region D of the (x, y)-plane where it possesses continuous first order partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$. Then a **tangent plane (linear) approximation** to f near any point (x_0, y_0) in D is given by

tangent plane approximation

$$z_{\mathrm{T}} = f(x_0, y_0) + (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0). \tag{35}$$

This linear expression approximates the function f close to the point (x_0, y_0) by a plane that is tangent to the surface $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$. The tangent plane approximation in (35) is an immediate extension to functions of two variables of the tangent line approximation in (34), to which it simplifies when only one independent variable is involved.

Both of these approximations are derived from the appropriate Taylor series expansions of functions discussed in Section 1.12 by retaining only the linear terms.

EXAMPLE 1.20

Find the tangent plane approximation to the function $z = x^2 - 3y^2$ near the point $(1, 2)$.

Solution Setting $x_0 = 1$, $y_0 = 2$ and substituting into (35) gives

$$z \approx -11 + 2(x-1) - 12(y-2) \text{ for } (x, y) \text{ close to } (1, 2).$$

1.11 Integrals

indefinite and definite integrals

A differentiable function $F(x)$ is called an **antiderivative** of the function $f(x)$ on some interval if at each point of the interval $dF/dx = f(x)$. If $F(x)$ is any antiderivative of $f(x)$, the **indefinite integral** of $f(x)$, written $\int f(x)\,dx$, is

$$\int f(x)\,dx = F(x) + c,$$

where c is an arbitrary constant called the *constant of integration*. The function $f(x)$ is called the **integrand** of the integral. Thus, an indefinite integral is a function, and an antiderivative and an indefinite integral can only differ by an arbitrary additive constant.

The expression $\int_a^b f(x)\,dx$, called a **definite integral**, is a number and may be interpreted geometrically as the area between the graph of $f(x)$ and the lines $x = a$ and $x = b$, for $b > a$, with areas above the x-axis counted as positive and those below it as negative.

The relationship between definite integrals that are *numbers* and indefinite integrals that are *functions* is given in the next theorem, included in which is also the mean value theorem for integrals. See the references at the end of the chapter for proofs and further information.

THEOREM 1.4

Fundamental theorem of integral calculus and the mean value theorem for integrals
If $F'(x)$ is continuous in the interval $a \le x \le b$, throughout which $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Another result is

$$\int_a^b f(x)\,dx = (b-a)\,f'(\xi),$$

if f is differentiable, where the number ξ, although unknown, lies in the interval $a < \xi < b$. In this form the result is called the **mean value theorem for integrals**. ■

An **improper integral** is a definite integral in which one or more of the following cases arises: (a) the integrand becomes infinite inside or at the end of the interval of integration, or (b) one (or both) of the limits of integration is infinite.

Types of Improper Integrals

Case (a)

convergence and divergence of improper integrals

If the integrand of an integral becomes infinite at a point c inside the interval of integration $a \leq x \leq b$ as shown in Fig. 1.12a, the improper integral is said to exist if the limits in (36) exist. When the improper integral exists it is said to **converge** to the (finite) value of the following limit:

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\int_a^{c-h} f(x)\,dx + \lim_{k\to 0}\int_{c+k}^b f(x)\,dx. \tag{36}$$

In this definition $h > 0$ and $k > 0$ are allowed to tend to zero *independently* of each other. If, when the limit is taken, the integral is either infinite or indeterminate, the integral is said to **diverge**.

Cauchy principal value

Some integrals of this type diverge when h and k are allowed to tend to zero independently of each other, but converge when the limit is taken with $h = k$, in which case the result of the limit is called the **Cauchy principal value** of the integral. Integrals of this type arise frequently when certain types of definite integral are evaluated in the complex plane by means of contour integration (see Chapter 15, Section 15.5).

Case (b)

If a limit of integration in a definite integral is infinite, say the upper limit as shown in Fig. 1.12b, then, when it exists, the improper integral is said to **converge** to the value of the limit

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx, \tag{37}$$

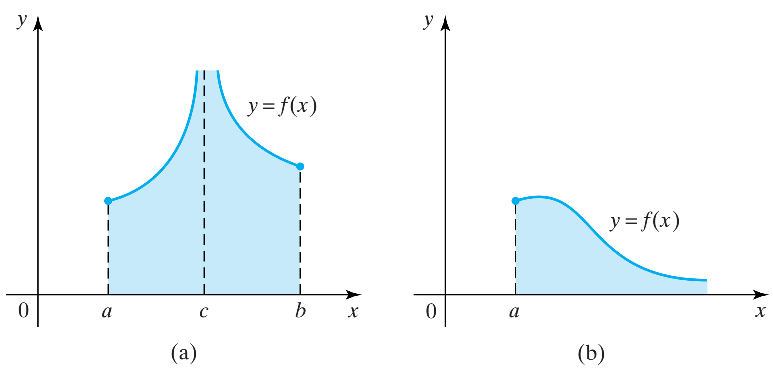


FIGURE 1.12 (a) $f(x)$ is infinite inside the interval of integration. (b) The interval of integration is infinite in length.

and the integral is **divergent** if the limit is either infinite or indeterminate. If both limits are infinite, the improper integral is said to **converge** to the value of the limit

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty, S\to\infty} \int_{-S}^{R} f(x)\,dx \tag{38}$$

when it exists, and the integral is said to be **divergent** if the limit is either infinite or indeterminate.

In (38) R and S are allowed to tend to infinity *independently* of each other. Integrals of this type also have Cauchy principal values if the foregoing process leads to divergence, but the integrals are convergent when the limit is taken with $R = S$. Integrals of this type also occur when certain real integrals are evaluated by means of contour integration (see Chapter 15, Section 15.5).

Elementary examples of convergent improper integrals of the types shown in (36) to (38) are

$$\int_0^1 \frac{x^p - x^{-p}}{x-1}\,dx = \frac{1}{p} - \pi \cot p\pi, \quad (p^2 < 1),$$

$$\int_0^{\infty} \exp(-x)\sin x dx = 1/2 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

THEOREM 1.5

Differentiation under the integral sign — Leibniz' rule If $\xi(t)$, $\eta(t)$, $d\xi/dt$, $d\eta/dt$, $f(x,t)$, and $\partial f/\partial t$ are continuous for $t_0 \le t \le t_1$ and for x in the interval of integration, then

$$\frac{d}{dt}\int_{\xi(t)}^{\eta(t)} f(x,t)\,dx = \int_{\xi(t)}^{\eta(t)} \frac{\partial f(x,t)}{\partial t}\,dx + f(\eta(t),t)\frac{d\eta}{dt} - f(\xi(t),t)\frac{d\xi}{dt}. \quad \blacksquare$$

This theorem is used, for example, in Chapter 18 when discussing discontinuous solutions of a class of partial differential equations called *conservation laws*. Extensions of the theorem to functions of more variables are developed in Chapter 12, Section 12.3, where certain vector integral theorems are developed, and applications of the results of that section to fluid mechanics are to be found in Chapter 12, Section 12.4.

An application of Theorem 1.5 that is easily checked by direct calculation is

$$\frac{d}{dt}\int_{2t}^{t^2} (x^2 + t)\,dx = \int_{2t}^{t^2} dx + (t^4 + t)\cdot 2t - (4t^2 + t)\cdot 2 = 2t^5 - 5t^2 - 4t.$$

A proof of Leibniz' rule can be found, for example, in Chapter 12 of reference [1.6].

1.12 Taylor and Maclaurin Theorems

THEOREM 1.6

Taylor's theorem for a function of one variable Let a function $f(x)$ have derivatives of all orders in the interval $a < x < b$. Then for each positive integer n and

each x_0 in the interval

$$f(x) = f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + R_{n+1}(x),$$

where $f^{(r)}(x) = d^r f/dx^r$, and the **remainder term** $R_{n+1}(x)$ is given by

$$R_{n+1}(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi),$$

for some ξ between x_0 and x. ■

Taylor polynomial

Maclaurin's theorem

Taylor's theorem becomes the **Taylor series** for $f(x)$ when n is allowed to become infinite, and if the remainder term is neglected in Taylor's theorem the result is called the **Taylor polynomial approximation** to $f(x)$ of **degree** n. The Taylor polynomial of degree 1 is simply the tangent line approximation to f at x_0 given in (34).

Taylor's theorem reduces to **Maclaurin's theorem** if $x_0 = 0$, and if we allow n to become infinite in Maclaurin's theorem, it becomes the **Maclaurin series** for $f(x)$.

A special case of Theorem 1.6 arises when Taylor's theorem is terminated with the term $R_1(x)$, corresponding to $n = 0$, because the result can be written

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(\xi), \tag{39}$$

mean value theorem

with ξ between x_0 and x, and in this form it is called the **mean value theorem for derivatives** (see the last result of Theorem 1.4).

A Taylor series is an example of an infinite series called a **power series**, the general form of which is

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots. \tag{40}$$

In (40) the quantity x is a variable, the numbers a_i are the **coefficients** of the power series, the constant x_0 is called the **center** of the series, or the point about which the series is **expanded**, and unless otherwise stated, x, x_0, and the a_i are real numbers, so the power series is a function of x.

A power series is said to **converge** for a given value of x if the sum of the infinite series for this value of x is *finite*. If the sum is *infinite*, or is *not defined*, the power series will be said to **diverge** for that value of x. Power series converge in an interval $x_0 - R < x < x_0 + R$, where the number R is called the **radius of convergence** of the series. Expressions for R are derived in Section 15.1.

The interval $x_0 - R < x < x_0 + R$ is called the **interval of convergence** of the power series. A power series converges for all x inside the interval of convergence and diverges for all x outside it, and the series may, or may not, converge at the end points of the interval. The convergence properties of power series are shown diagramatically in Fig. 1.13, and results (40) and combining expressions for R with

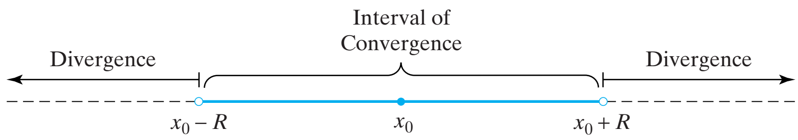


FIGURE 1.13 Interval of convergence of a power series with center x_0.

(40) gives the following theorem (see the references at the end of the chapter for real variable proofs of the following results and for more information).

THEOREM 1.7

Ratio test and *n*th root test for the convergence of power series The power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

radius and interval of convergence

converges in the *interval of convergence* $x_0 - R < x < x_0 + R$, where the *radius of convergence* R is determined by either of the formulas

$$\text{(a) } R = 1/\lim_{n\to\infty} |a_{n+1}/a_n| \quad \text{or} \quad \text{(b) } R = 1/\lim_{n\to\infty} |a_n|^{1/n}.$$

The power series will diverge outside the interval of convergence, and its behavior at the ends of the interval of convergence must be determined separately. ■

A simple result on the convergence of a series that is often useful is the alternating series test. An **alternating series** is so named because the signs of successive terms of the series alternate in sign.

THEOREM 1.8

The alternating series test for convergence The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if $a_n > 0$ and $a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$. ■

The following theorem on the differentiation and integration of power series is often needed, and it is a real variable form of a result proved later in Chapter 15 when complex power series are studied.

THEOREM 1.9

Differentiation and integration of power series Let a power series have an interval of convergence $x_0 - R < x < x_0 + R$. Then the series may be differentiated and integrated term by term, and in each case the resulting series will have the same interval of convergence as the original series. In addition, within an interval of convergence common to any two power series, the series may be scaled by a constant and added or subtracted term by term and the resulting power series will have the same common interval of convergence. ■

The simplest form of Taylor's theorem for a function of two variables that finds many applications is given in the next theorem.

THEOREM 1.10

Taylor's theorem for a function of two variables Let $f(x, y)$ be defined for $a < x < b$ and $c < y < d$ and have continuous partial derivatives up to and including

those of order 2. Then for x_0 and y_0 any points such that $a < x_0 < b$ and $c < y_0 < d$,

$$\begin{aligned} f(x, y) = {} & f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) \\ & + \frac{1}{2!}\big[(x - x_0)^2 f_{xx}(x_0 + \xi, y_0 + \eta) + 2(x - x_0)(y - y_0) \\ & \times f_{xy}(x_0 + \xi, y_0 + \eta)(y - y_0)^2 f_{yy}(x_0 + \xi, y_0 + \eta)\big], \end{aligned}$$

where the numbers ξ and η are unknown, but ξ lies between x_0 and x and η lies between y_0 and y. ■

The group of second order partial derivatives in Theorem 1.10 forms the remainder term, and when these derivatives are ignored, the result reduces to the tangent plane approximation to $f(x, y)$ at the point (x_0, y_0) given in (35).

More information on Taylor's theorem and series can be found, for example, in reference [1.2].

1.13 Cylindrical and Spherical Polar Coordinates and Change of Variables in Partial Differentiation

Mathematical problems formulated using a particular coordinate system, such as cartesian coordinates, often need to be reexpressed in terms of a different coordinate system in order to simplify the task of finding a solution. When partial derivatives occur in the formulation of problems, it becomes necessary to know how they transform when a different coordinate system is used. The fundamental theorem governing the transformation of partial derivatives under a change of variables takes the following form (see the references at the end of the chapter for the proof of Theorem 1.11 and for more examples of its use).

THEOREM 1.11 **Change of variables in partial differentiation** Let $f(x_1, x_2, \ldots, x_n)$ be a differentiable function with respect to the n independent variables $x_1, x_2, \ldots, x_n$, and let the n new independent variables $u_1, u_2, \ldots, u_n$ be determined in terms of $x_1, x_2, \ldots, x_n$ by

$$x_1 = X_1(u_1, u_2, \ldots, u_n), \quad x_2 = X_2(u_1, u_2, \ldots, u_n), \ldots, \quad x_n = X_n(u_1, u_2, \ldots, u_n),$$

where $X_1, X_2, \ldots, X_n$ are differentiable functions of their arguments. Then, if as a result of the change of variables the function $f(x_1, x_2, \ldots, x_n)$ becomes the function $F(X_1, X_2, \ldots, X_n)$, and using chain rules we have

$$\begin{aligned} \frac{\partial F}{\partial u_1} &= \frac{\partial f}{\partial x_1}\frac{\partial X_1}{\partial u_1} + \frac{\partial f}{\partial x_2}\frac{\partial X_2}{\partial u_1} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial X_n}{\partial u_1} \\ \frac{\partial F}{\partial u_2} &= \frac{\partial f}{\partial x_1}\frac{\partial X_1}{\partial u_2} + \frac{\partial f}{\partial x_2}\frac{\partial X_2}{\partial u_2} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial X_n}{\partial u_2} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial F}{\partial u_n} &= \frac{\partial f}{\partial x_1}\frac{\partial X_1}{\partial u_n} + \frac{\partial f}{\partial x_2}\frac{\partial X_2}{\partial u_n} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial X_n}{\partial u_n}. \end{aligned} \tag{41}$$

■

To find higher order partial derivatives it is necessary to express the relationships between the *operations* of differentiation in the two coordinate systems, rather than between the actual derivatives themseves. This can be accomplished by rewriting the results of Theorem 1.11 in the form of **partial differential operators** as follows:

$$\begin{aligned}
\frac{\partial}{\partial u_1} &\equiv \frac{\partial X_1}{\partial u_1}\frac{\partial}{\partial x_1} + \frac{\partial X_2}{\partial u_1}\frac{\partial}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial u_1}\frac{\partial}{\partial x_n} \\
\frac{\partial}{\partial u_2} &\equiv \frac{\partial X_1}{\partial u_2}\frac{\partial}{\partial x_1} + \frac{\partial X_2}{\partial u_2}\frac{\partial}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial u_2}\frac{\partial}{\partial x_n} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial}{\partial u_n} &\equiv \frac{\partial X_1}{\partial u_n}\frac{\partial}{\partial x_1} + \frac{\partial X_2}{\partial u_n}\frac{\partial}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial u_n}\frac{\partial}{\partial x_n}.
\end{aligned} \tag{42}$$

When expressed in this form the relationships between the partial differentiation operations $\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$ and $\partial/\partial u_1, \partial/\partial u_2, \ldots, \partial/\partial u_n$ become clear. This interpretation is needed when finding higher order partial derivatives such as $\partial^2 F/\partial u_2 \partial u_1$, because

$$\frac{\partial^2 F}{\partial u_2 \partial u_1} = \frac{\partial}{\partial u_1}\left(\frac{\partial F}{\partial u_2}\right) = \left(\frac{\partial X_1}{\partial u_1}\frac{\partial}{\partial x_1} + \frac{\partial X_2}{\partial u_1}\frac{\partial}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial u_1}\frac{\partial}{\partial x_n}\right)\left(\frac{\partial F}{\partial u_2}\right).$$

An important combination of partial derivatives that occurs throughout physics and engineering is called the **Laplacian** of a function. When a twice differentiable function $f(x, y, z)$ of the cartesian coordinates x, y, and z is involved, the Laplacian of f, denoted by Δf and sometimes by $\nabla^2 f$, read "del squared f," takes the form

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{43}$$

Cylindrical Polar Coordinates (r, θ, z)

The cylindrical polar coordinate system (r, θ, z) is illustrated in Fig. 1.14, and its relationship to cartesian coordinates is given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad \text{with } 0 \le \theta < 2\pi \text{ and } r \ge 0. \tag{44}$$

Spherical Polar Coordinates (r, ϕ, θ)

The spherical polar coordinate system (r, ϕ, θ) shown in Fig. 1.15 is related to cartesian coordinates by

$$\begin{aligned} x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \\ \text{with } 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi. \end{aligned} \tag{45}$$

The derivation of the formulas for the change of variables in functions of several variables can be found in any one of references [1.1] to [1.7], where cylindrical and

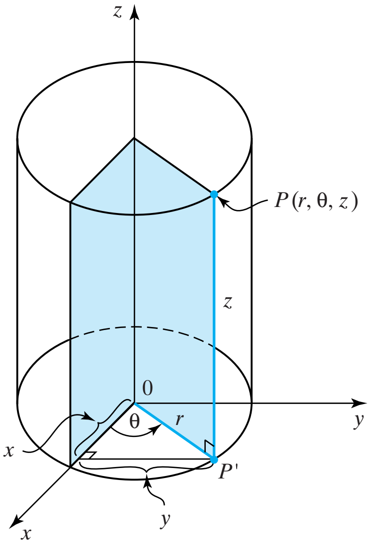


FIGURE 1.14 Cylindrical polar coordinates (r, θ, z).

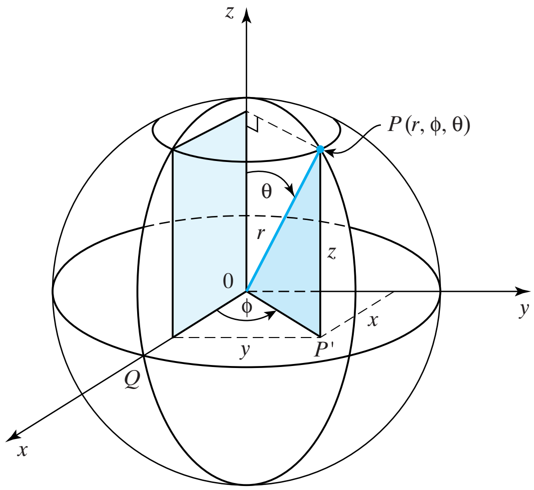


FIGURE 1.15 Spherical polar coordinates (r, ϕ, θ).

spherical polar coordinates are also discussed. Information on general orthogonal coordinate systems can be found in references [G.3] and [2.3].

EXERCISES 1.13

1. By making the change of variables $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, in the function $f(x, y, z)$, when it becomes the function $F(r, \theta, z)$, show that in cylindrical polar coordinates

$$\frac{\partial F}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y},$$
$$\frac{\partial F}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}, \quad \frac{\partial F}{\partial z} = \frac{\partial f}{\partial z}.$$

2. Use the results of Exercise 1 to show that in cylindrical polar coordinates the Laplacian

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{becomes}$$
$$\Delta F = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \frac{\partial^2 F}{\partial z^2},$$

and hence that an equivalent form of ΔF is

$$\Delta F = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial F}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial F}{\partial z}\right)\right].$$

3. By making the change of variable $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ in the function $f(x, y, z)$, when it becomes $F(r, \phi, \theta)$, show that in spherical polar coordinates

$$\frac{\partial F}{\partial r} = \sin\theta\cos\phi\frac{\partial f}{\partial x} + \sin\phi\sin\theta\frac{\partial f}{\partial y} + \cos\phi\frac{\partial f}{\partial z}$$
$$\frac{\partial F}{\partial \phi} = r\cos\phi\cos\theta\frac{\partial f}{\partial x} + r\cos\phi\sin\theta\frac{\partial f}{\partial y} - r\sin\phi\frac{\partial f}{\partial z}$$
$$\frac{\partial F}{\partial z} = -r\sin\phi\sin\theta\frac{\partial f}{\partial x} + r\sin\phi\cos\theta\frac{\partial f}{\partial y}.$$

4. Use the results of Exercise 3 to show that in spherical polar coordinates the Laplacian

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

becomes

$$\Delta F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 F}{\partial \phi^2}\right)$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial F}{\partial \theta}\right).$$

1.14 Inverse Functions and the Inverse Function Theorem

In mathematics and its applications it is often necessary to find the inverse of a function $y = f(x)$ so x can be expressed in the form $x = g(y)$, and when this can be done the function g is called the **inverse** of f and is such that $y = f(g(y))$. When f is an arbitrary function its inverse is often denoted by f^{-1}, and this superscript notation is also used to denote the inverse of trigonometric functions so if, for example, $y = \sin x$, the inverse sine function is written $\sin^{-1}$, so that $x = \sin^{-1} y$. However, the notation $y = \arcsin y$ is also used with the understanding that the notations arcsin and $\sin^{-1}$ are equivalent.

A trivial example of a function whose inverse can be found unambiguously is $y = ax + b$, because provided $a \neq 0$ we can write $x = (y - b)/a$ for all x and y. This is not the case, however, when trigonometric functions are involved, because the function $y = \sin x$ will give a unique value of y for any given x, but given y there are infinitely many values of x for which $y = \sin x$. This and similar inverse trigonometric functions are considered in elementary calculus courses. There the multivalued nature of the inverse sine function is resolved by restricting it to make y lie in a specific interval chosen so that one y corresponds to one x and, conversely, one x corresponds to one y. This situation is described by saying that the relationship between x and y is **one-to-one**. Specifically, in the case of the sine function, this is accomplished by requiring that if $x = \sin y$, the inverse function $y = \text{Arcsin } x$ is restricted so its **principal value** lies in the interval $-\pi/2 \leq \text{Arcsin} x \leq \pi/2$, where the domain of definition of the inverse function is $-1 \leq x \leq 1$.

A different possibility that arises frequently is when x and y are related by an equation of the form $f(x, y) = 0$ from which it is impossible to extract either x as a function of y, or y as a function of x in terms of known functions. A typical example of this type is $f(x, y) = x^2 - 2y^2 - \sin xy$. To make matters precise, if x and y are related by an equation $f(x, y) = 0$, then if a function $y = g(x)$ exists such that $f(x, g(x)) = 0$, the function $y = g(x)$ is said to be defined **implicitly** by $f(x, y) = 0$.

Although it is often not possible to find the function $g(x)$, it is still necessary to know when, in a neighborhood of a point (x_0, y_0), given a value of x, a unique value of y can be found, sometimes only numerically. The *implicit function theorem* that follows is seldom mentioned in first calculus courses because its proof involves certain technicalities, but it is quoted here in the simplest possible form because of its fundamental importance and the fact that is it frequently used by implication.

THEOREM 1.12

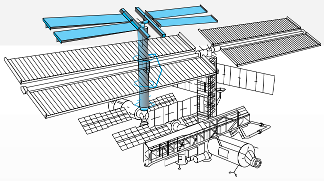
The implicit function theorem Let $f(x, y)$ and $f_y(x, y)$ be continuous in a region D of the (x, y)-plane and let (x_0, y_0) be a point inside D, where $f(x_0, y_0) = 0$ and $f_y(x_0, y_0) \neq 0$. Then

(i) There is a rectangle R inside D containing (x_0, y_0) at all points of which there can be found a unique y such that $f(x, y) = 0$.

(ii) If the value of y is denoted by $g(x)$, then $y_0 = g(x_0)$, with $f(x, g(x)) = 0$, and $g(x)$ is continuous inside R.

(iii) If, in addition, $f_x(x, y)$ is continuous in D then $g(x)$ is differentiable in R and $g'(x) = -\frac{f_x(x,g(x))}{f_y(x,g(x))}$. ■

In general terms, the implicit function theorem gives conditions that ensure the existence of an inverse function that is continuous and smooth enough to be differentiable. The theorem has a more general form involving functions $f(x_1, x_2, \ldots, x_n)$ of n variables, though this will not be given here. The interested reader can find accounts of the implicit function theorem and some of its generalizations in references [1.4], [1.6], and [5.1].



CHAPTER 1
TECHNOLOGY PROJECTS

Project 1

Linear Difference Equations and the Fibonacci Sequence

In Italy in 1202, Leonardo of Pisa, also known as Fibonacci, posed the following question. Let a newly born pair of rabbits produce two offspring each month, with breeding starting when they are 2 months old. Assuming that the pair of offspring start breeding in the same fashion when 2 months old, and that the process continues thereafter in a similar manner with no deaths, how many pairs of rabbits will there be after n months?

If u_n, is the number of pairs of rabbits after n months, the production of rabbits can be represented by the **linear difference equation**, or **recurrence relation**,

$$u_{n+2} = u_{n+1} + u_n,$$

where the sequence of numbers u_r with $r = 1, 2, \ldots$ is generated by setting $u_1 = 1$ and $u_2 = 1$, since this represents the initial pair of rabbits that began the breeding process. A simple calculation using this difference equation shows that the sequence of numbers generated in this manner that represents the number of pairs of rabbits present each month is

$$1, 1, 2, 3, 5, 8, \ldots,$$

and this is called the **Fibonacci** sequence. This sequence is found to occur in the study of regular solids, in numerical analysis, and elsewhere in mathematics.

A linear difference equation of the form

$$u_{n+2} = au_{n+1} + bu_n,$$

with a and b real numbers, can be solved by substituting $u_n = A\lambda^n$ into the difference equation and finding the two roots λ_1 and λ_2 of the resulting quadratic equation in λ. When $\lambda_1 \neq \lambda_2$, the general solution is $u_n = A_1\lambda_1^n + A\lambda_2^n$, and when $\lambda_1 = \lambda_2 = \lambda$, say, the general solution is $u_n = (A_1 + nA_2)\mu^n$. The arbitrary constants A_1 and A_2 are found by requiring u_n to satisfy some given conditions of the form $u_1 = \alpha$ and $u_2 = \beta$, where the numbers α and β specify the way the sequence starts (the **initial conditions**).

Use this method to show that the solution u_n for the Fibonacci sequence is

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad \text{for } n = 1, 2, \ldots.$$

Make use of computer algebra to generate the first 30 terms of the Fibonacci sequence directly from the difference equation, and verify that the results are in agreement with the preceding formula.

Use computer algebra to show that $\lim_{n\to\infty}(u_n/u_{n-1}) = \frac{1}{2}(\sqrt{5}+1)$. This number is called the **golden mean**, and in art and architecture it represents the ratio of the sides of a rectangle that is considered to have the most pleasing appearance.

Project 2

Erratic Behavior of a Sequence Generated by a Difference Equation

1. Not all difference equations generate sequences of numbers that evolve steadily as happens with the Fibonacci sequence. Use computer algebra to generate the first 20 terms of the sequence produced by the difference equation

$$u_{n+2} = 2u_{n+1} - 5u_n \quad \text{with } u_1 = 1, u_2 = -3,$$

and observe its erratic behavior. Use the method of Project 1 to determine the analytical solution, and by means of computer algebra confirm that the two results are in agreement. Examine the analytical solution and explain why the behavior of the sequence of terms is so erratic.

2. Construct a difference equation of your own in which the roots λ_1 and λ_2 are equal. Find the analytical solution and use computer algebra to determine the first 20 terms of the sequence. Verify that these terms are in agreement with the ones generated directly from the difference equation.

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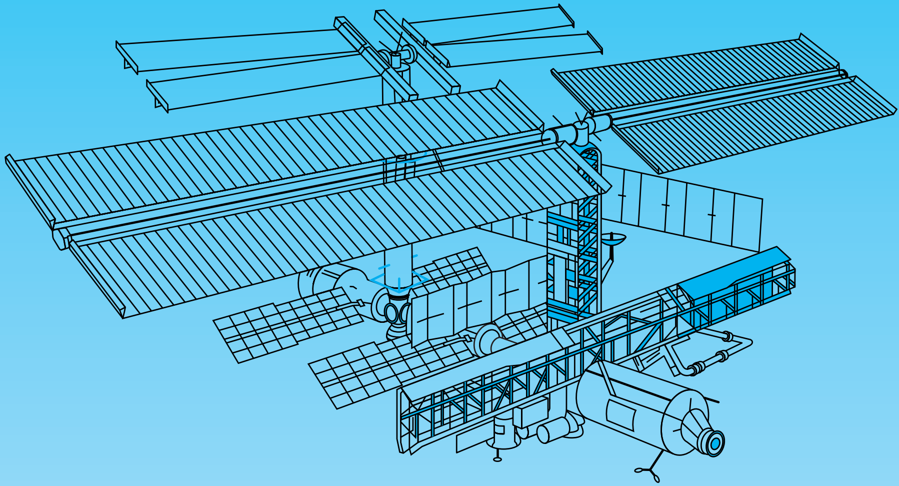
PART TWO

VECTORS AND MATRICES



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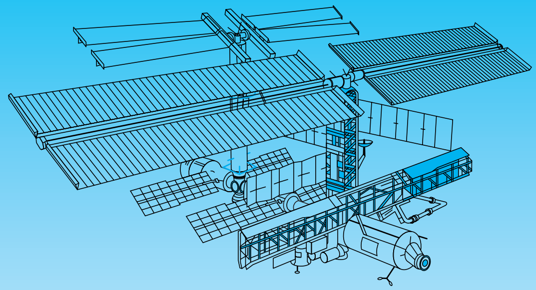
CHAPTER 2



Vectors and Vector Spaces

Engineers, scientists, and physicists need to work with systems involving physical quantities that, unlike the density of a solid, cannot be characterized by a single number. This chapter is about the algebra of important and useful quantities called vectors that arise naturally when studying physical systems, and are defined by an ordered group of three numbers (a, b, c). Vectors are of fundamental importance and they play an essential role when the laws governing engineering and physics are expressed in mathematical terms.

A scalar quantity is one that is completely described when its magnitude is known, such as pressure, temperature, and area. A vector is a quantity that is completely specified when both its magnitude and direction are given, such as force, velocity, and momentum. A vector can be described geometrically as a directed straight line segment, with its length proportional to the magnitude of the vector, the line representing the vector parallel to the line of action of the vector, and an arrow on the line showing the direction along the line, or the sense, in which the vector acts.

This geometrical interpretation of a vector is valuable in many ways, as it can be used to add and subtract vectors and to multiply them by a scalar, since this merely involves changing their magnitude and sense, while leaving the line to which they are parallel unchanged. However, to perform more general algebraic operations on vectors some other form of representation is required. The one that is used most frequently involves describing a vector in terms of what are called its components along a set of three mutually orthogonal axes, which are usually taken to be the axes O$\{x, y, z\}$ in the cartesian coordinate system. Here, by the component of a vector along a given line l, we mean the length of the perpendicular projection of the vector onto the line l.

We will see later that this cartesian representation of a vector identifies it completely in terms of three components and enables algebraic operations to be performed on it. In particular, it allows the introduction of the scalar product, or dot product, of two vectors that results in a scalar, and a vector product, or cross product, of two vectors that leads to a vector.

Finally, vectors and their algebra will be generalized to n space dimensions, leading to the concept of a vector space and to some related ideas.

2.1 Vectors, Geometry, and Algebra

Many quantities are completely described once their magnitude is known. A typical example of a physical quantity of this type is provided by the temperature at a given point in a room that is determined by the number specifying its value measured on a temperature scale, such as degrees F or degrees C. A quantity such as this is called a **scalar** quantity, and different examples of mathematical and physical scalar quantities are real numbers, length, area, volume, mass, speed, pressure, chemical concentration, electrical resistance, electric potential, and energy.

scalar

Other physical quantities are only fully specified when both their magnitude and direction are given. Quantities like this are called **vector** quantities, and a typical example of a vector quantity arises when specifying the instantaneous motion of a fluid particle in a river. In this case both the particle speed and its direction must be given if the description of its motion is to be complete. Speed in a given direction is called **velocity**, and velocity is a vector quantity. Some other examples of vector quantities are force, acceleration, momentum, the heat flow vector at a point in a block of metal, the earth's magnetic field at a given location, and a mathematical quantity called the gradient of a scalar function of position that will be defined later. By definition, the magnitude of a vector quantity is a nonnegative number (a scalar) that measures its size without regard to its direction, so, for example, the magnitude of a velocity is a speed.

vector

A convenient geometrical representation of a vector is provided by a straight line segment drawn in space parallel to the required direction, with an arrowhead indicating the **sense** in which the vector acts along the line segment, and the length of the line segment proportional to the magnitude of the vector. This is called a **directed straight line segment**, and by definition all directed straight line segments that are parallel to one another and have the same sense and length are regarded as equal. Expressed differently, moving a directed straight line segment parallel to itself so that its length remains the same and its arrow still points in the same direction leaves the vector it represents unchanged. A shift of a directed straight line segment of this type is called a **translation** of the vector it represents. For this reason the terms *directed straight line segment* and *vector* can be used interchangeably. Some examples of vectors that are equal through translation are shown in Fig. 2.1.

directed straight line segment

translation

It must be emphasized that geometrical representations of vectors as directed straight line segments in space are defined without reference to a specific coordinate system. This purely geometrical interpretation of vectors finds many applications, though a different form of representation is necessary if an effective vector algebra is to be developed for use with the calculus. An analytical representation of vectors that allows a vector algebra to be constructed with this purpose in mind can be based on a general coordinate system. However, throughout this chapter only rectangular cartesian coordinates will be used because they provide a simple and natural way of representing vectors.

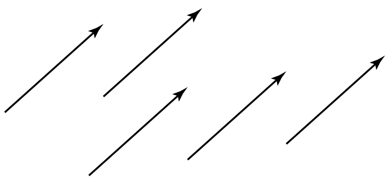


FIGURE 2.1 Equal geometrical vectors.

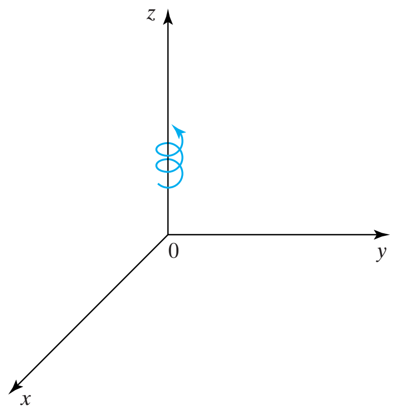


FIGURE 2.2 A right-handed rectangular cartesian coordinate system.

In rectangular cartesian coordinates the x-, y-, and z-axes are all mutually orthogonal (perpendicular), and the positive sense along the axes is taken to be in the direction of increasing x, y, and z. The orientation of the axes will always be such that the positive direction along the z-axis is the one in which a right-handed screw (such as a corkscrew) aligned with the z-axis will advance when rotated from the positive x-axis to the positive y-axis, as shown in Fig. 2.2. A system of axes with this property is called a **right-handed system**.

right-handed system

The end of a vector toward which the arrow points will be called the **tip** of the vector, and the other end its **base**. Because a vector is invariant under a translation, there is no loss of generality in taking its base to be located at the origin O of the coordinate system, and its tip at a point P with the coordinates (a_1, a_2, a_3), say, as shown in Fig. 2.3. An application of the Pythagoras theorem to the triangle OPP'

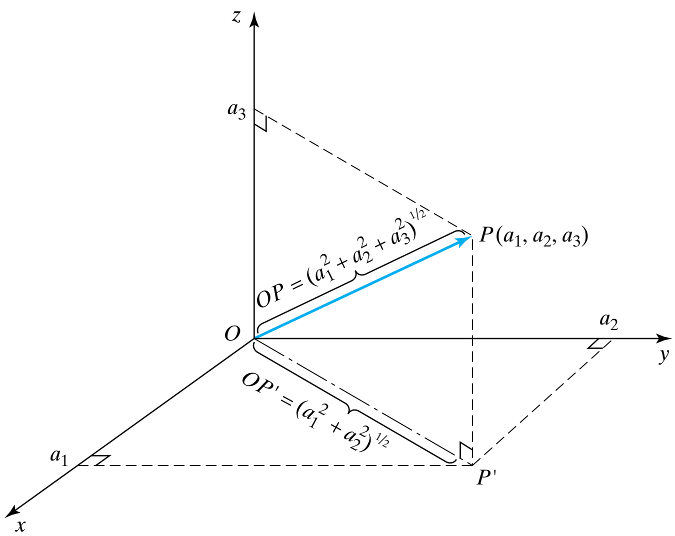


FIGURE 2.3 The vector from O to P and its components a_1, a_2, and a_3 in the x-, y-, z-coordinate system.

magnitude, unit vector, and components

shows the length of the line from O to P to be $(a_1^2 + a_2^2 + a_3^2)^{1/2}$. This length is *proportional* to the **magnitude** of the vector it represents, and as the base of the vector is at O, the sense of the vector is from O to P. For convenience, the constant of proportionality will be taken to be 1, so a directed straight line segment of unit length will represent a vector of magnitude 1 and so will be called a **unit vector**. Using this convention, the vector represented by the line from O to P in Fig. 2.3 has magnitude $(a_1^2 + a_2^2 + a_3^2)^{1/2}$. The three numbers a_1, a_2, and a_3, in this order, that define the vector from O to P are called its **components** in the x, y, and z directions, respectively.

ordered number triple

A set of three numbers a_1, a_2, and a_3 in a given order, written (a_1, a_2, a_3), is called an **ordered number triple**. As the coordinates (a_1, a_2, a_3) of point P in Fig. 2.3 completely define the vector from O to P, this ordered number triple may be taken as the definition of the vector itself. In general, changing the order of the numbers in an ordered number triple changes the vector it defines.

Sometimes it is necessary to consider a vector whose base does not coincide with the origin. Suppose that when this occurs the base C is at the point (c_1, c_2, c_3) and the tip D is at the point (d_1, d_2, d_3). Then Fig. 2.4 shows the components of this vector in the x, y, and z directions to be $d_1 - c_1$, $d_2 - c_2$, and $d_3 - c_3$. These components determine both the magnitude and direction of the vector. The vector is described by the ordered number triple $(d_1 - c_1, d_2 - c_2, d_3 - c_3)$, and the length of CD that is equal to the magnitude of the vector is $[(d_1 - c_1)^2 + (d_2 - c_2)^2 + (d_3 - c_3)^2]^{1/2}$.

norm and modulus

For convenience, it is usual to represent a vector by a single boldface character such as $\mathbf{a}$, and its **magnitude** (length) by $\|\mathbf{a}\|$, called the **norm** of $\mathbf{a}$. It is necessary to say here that in applications of vectors to mechanics, and in some purely geometrical applications of vectors, the norm of vector $\mathbf{r}$ is often called its **modulus** and written $|\mathbf{r}|$. When this convention is used, because $|\mathbf{r}|$ is a scalar it is usual to denote it by the corresponding ordinary italic letter r, so that $r = |\mathbf{r}|$.

If the base and tip of a vector need to be identified by letters, a vector such as the one from C to D in Fig. 2.4 is written $\underline{CD}$, with underlining used to indicate that a vector is involved, and the ordering of the letters is such that the first shows the

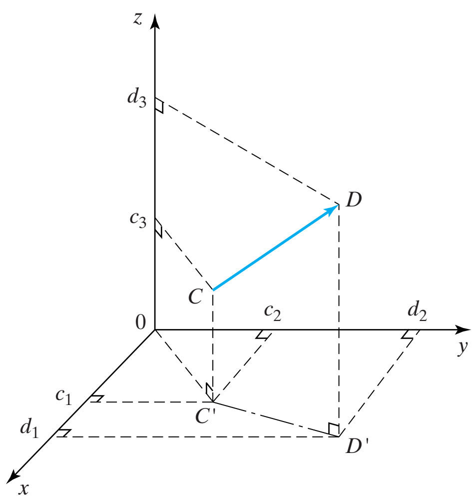


FIGURE 2.4 Vector directed from point C at (c_1, c_2, c_3) to point D at (d_1, d_2, d_3).

base and the second the tip of the vector. Thus, $\underline{CD}$ and $\underline{DC}$ are vectors of equal magnitude but opposite sense, and when these vectors are represented by arrows, the arrows are parallel and of equal length, but point in opposite directions.

EXAMPLE 2.1

If, in Fig. 2.4, C is the point $(-3, 4, 9)$ and D the point $(2, 5, 7)$, the vector $\underline{CD}$ has components $2-(-3)=5$, $5-4=1$, and $7-9=-2$, and so is represented by the ordered number triple $(5, 1, -2)$, whereas vector $\underline{DC}$ has components -5, -1, and 2 and is represented by the ordered number triple $(-5, -1, 2)$. ■

Having illustrated the concepts of scalars and vectors using some familiar examples, we now develop the algebra of vectors in rather more general terms.

Vectors

A **vector** quantity $\mathbf{a}$ is an ordered number triple (a_1, a_2, a_3) in which a_1, a_2, and a_3 are real numbers, and we shall write $\mathbf{a} = (a_1, a_2, a_3)$. The numbers a_1, a_2, and a_3, in this order, are called the first, second, and third **components** of vector $\mathbf{a}$ or, equivalently, its x-, y-, and z-components.

Null vector

The **null (zero)** vector, written $\mathbf{0}$, has neither magnitude nor direction and is the ordered number triple $\mathbf{0} = (0, 0, 0)$.

Equality of vectors

Two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are **equal**, written $\mathbf{a} = \mathbf{b}$, if, and only if, $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

EXAMPLE 2.2

If $\mathbf{a} = (a_1, -5, 6)$, $\mathbf{b} = (3, b_2, b_3)$ and $\mathbf{c} = (3, -5, 1)$, then $\mathbf{a} = \mathbf{b}$ if $a_1 = 3$, $b_2 = -5$ and $b_3 = 6$, and $\mathbf{b} = \mathbf{c}$ if $b_2 = -5$ and $b_3 = 1$, but $\mathbf{a} \neq \mathbf{c}$ for any choice of a_1 because $6 \neq 1$. ■

Norm of a vector

The **norm** of vector $\mathbf{a} = (a_1, a_2, a_3)$, denoted by $\|\mathbf{a}\|$, is the non-negative real number

$$\|\mathbf{a}\| = \left(a_1^2 + a_2^2 + a_3^2\right)^{1/2},$$

and in geometrical terms $\|\mathbf{a}\|$ is the *length* of vector $\mathbf{a}$. The norm of the null vector $\mathbf{0}$ is $\|\mathbf{0}\| = 0$. For example, if $\mathbf{a}$ is in m/sec, "length" of $\mathbf{a}$ is in m/sec.

EXAMPLE 2.3

If $\mathbf{a} = (1, -3, 2)$, then $\|\mathbf{a}\| = [1^2 + (-3)^2 + 2^2]^{1/2} = \sqrt{14}$, as illustrated in Fig. 2.5. ■

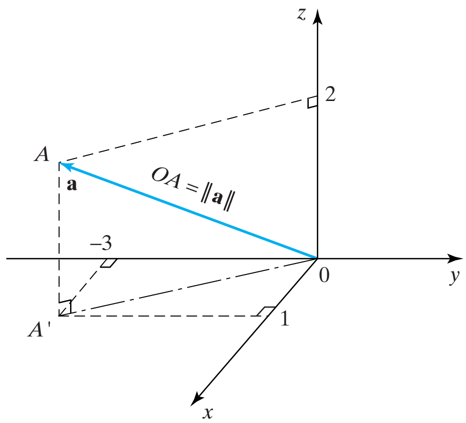


FIGURE 2.5 Vector **a** and its norm $\|\mathbf{a}\|$.

The sum of two vectors

If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ have the same dimensions, say, both are m/sec, their **sum**, written $\mathbf{a} + \mathbf{b}$, is defined as the ordered number triple (vector) obtained by adding corresponding components of **a** and **b** to give

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

EXAMPLE 2.4 If $\mathbf{a} = (1, 2, -5)$ and $\mathbf{b} = (-2, 2, 4)$, then

$$\mathbf{a} + \mathbf{b} = (1 + (-2), 2 + 2, -5 + 4) = (-1, 4, -1).$$

Multiplying a vector by a scalar

Let $\mathbf{a} = (a_1, a_2, a_3)$ and λ be an arbitrary real number. Then the product $\lambda\mathbf{a}$ is defined as the vector

$$\lambda\mathbf{a} = (\lambda a_1, \lambda a_2, \lambda a_3).$$

EXAMPLE 2.5 Let $\mathbf{a} = (2, -3, 5)$, $\mathbf{b} = (-1, 2, 4)$. Then $2\mathbf{a} = (4, -6, 10)$, $4\mathbf{b} = (-4, 8, 16)$, and $2\mathbf{a} + 4\mathbf{b} = (4 + (-4), -6 + 8, 10 + 16) = (0, 2, 26)$.

This definition of the product of a vector and a scalar, called **scaling** a vector, shows that when vector **a** is multiplied by a scalar λ, the norm of **a** is multiplied by $|\lambda|$, because

$$\|\lambda\mathbf{a}\| = \left(\lambda^2 a_1^2 + \lambda^2 a_2^2 + \lambda^2 a_3^2\right)^{1/2} = |\lambda| \cdot \|\mathbf{a}\|.$$

It also follows from the definition that the sense of vector **a** is reversed when it is multiplied by -1, though its norm is left unaltered. The definition of the **difference**

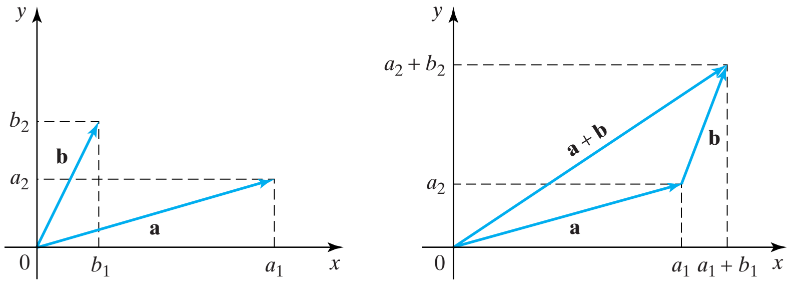


FIGURE 2.6 The vector sum $\mathbf{a} + \mathbf{b}$.

of two vectors is seen to be contained in the definition of their sum, because $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$. In particular, when $\mathbf{a} = \mathbf{b}$, we find that that $\mathbf{a} - \mathbf{a} = \mathbf{0}$, showing that $-\mathbf{a}$ is the *additive inverse* of $\mathbf{a}$.

The geometrical interpretations of the sum $\mathbf{a} + \mathbf{b}$, the difference $\mathbf{a} - \mathbf{b}$, and the **scaled** vector $\lambda\mathbf{a}$ in terms of their components are shown in Figs. 2.6 to 2.8, though to simplify the diagrams only the two-dimensional cases are illustrated. This involves no loss of generality, because it is always possible to choose the (x, y)-plane to coincide with the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$.

Vector Addition by the Triangle Rule

Consideration of Fig. 2.6 shows that the addition of vector $\mathbf{b}$ to vector $\mathbf{a}$ is obtained geometrically by translating vector $\mathbf{b}$ until its base is located at the tip of vector $\mathbf{a}$, and then the vector representing the sum $\mathbf{a} + \mathbf{b}$ has its base at the base of vector $\mathbf{a}$ and its tip at the tip of the repositioned vector $\mathbf{b}$. Because of the triangle involving vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} + \mathbf{b}$, this geometrical interpretation of a vector sum is called the **triangle rule** for vector addition. The triangle rule also applies to the difference of two vectors, as may be seen by considering Fig. 2.7, because after obtaining $-\mathbf{b}$ from $\mathbf{b}$ by reversing its sense, the difference $\mathbf{a} - \mathbf{b}$ can be written as the vector sum $\mathbf{a} + (-\mathbf{b})$, where $-\mathbf{b}$ is added to vector $\mathbf{a}$ by means of the triangle rule.

triangle rule for addition

The algebraic results discussed so far concerning the addition and scaling of vectors, together with some of their consequences, are combined to form the following theorem.

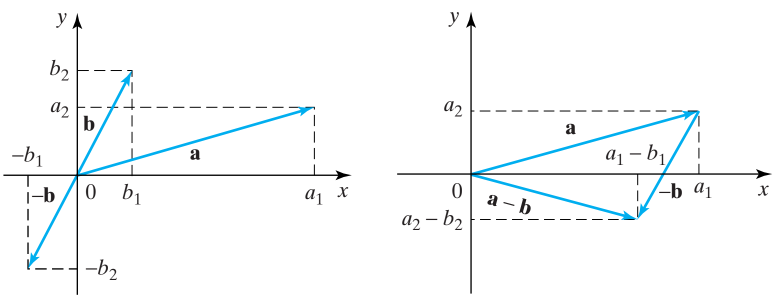


FIGURE 2.7 The vector difference $\mathbf{a} - \mathbf{b}$.

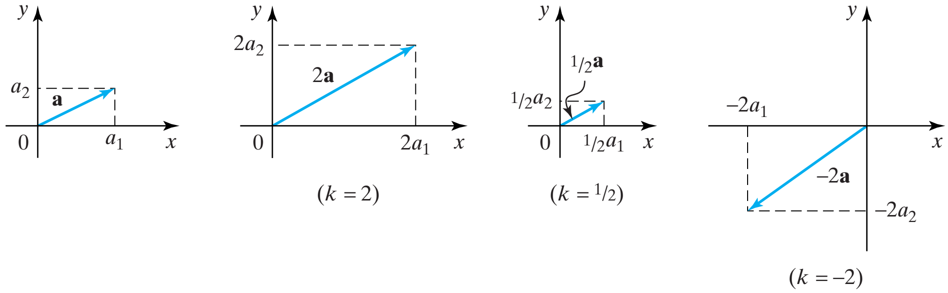


FIGURE 2.8 The vector $k\mathbf{a}$ for different values of k.

THEOREM 2.1

Addition and scaling of vectors Let $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ be arbitrary vectors and let α and β be arbitrary real numbers. Then:

1. $\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}$ (vector addition is **commutative**);
2. $\mathbf{P} + \mathbf{0} = \mathbf{0} + \mathbf{P} = \mathbf{P}$ (**0** is the **identity element** in vector addition);
3. $(\mathbf{P} + \mathbf{Q}) + \mathbf{R} = \mathbf{P} + (\mathbf{Q} + \mathbf{R})$ (vector addition is **associative**);
4. $\alpha(\mathbf{P} + \mathbf{Q}) = \alpha\mathbf{P} + \alpha\mathbf{Q}$ (multiplication by a scalar is **distributive** over **vector addition**);
5. $(\alpha\beta)\mathbf{P} = \alpha(\beta\mathbf{P}) = \beta(\alpha\mathbf{P})$ (multiplication of a vector by a product of scalars is **associative**);
6. $(\alpha + \beta)\mathbf{P} = \alpha\mathbf{P} + \beta\mathbf{P}$ (multiplication of a vector by a sum of scalars is **distributive**);
7. $\|\alpha\mathbf{P}\| = |\alpha| \cdot \|\mathbf{P}\|$ (scaling $\mathbf{P}$ by α scales the norm of $\mathbf{P}$ by $|\alpha|$).

Proof The results of this theorem are all immediate consequences of the above definitions so as the proofs of results 1 to 6 are all very similar, and result 7 has already been established, we only prove result 4.

$$
\begin{aligned}
\text{Let } \mathbf{P} &= (p_1, p_2, p_3) \text{ and } \mathbf{Q} = (q_1, q_2, q_3);\ \text{then} \\
\alpha(\mathbf{P} + \mathbf{Q}) &= \alpha(p_1 + q_1, p_2 + q_2, p_3 + q_3) \\
&= \alpha[(p_1, p_2, p_3) + (q_1, q_2, q_3)] \\
&= \alpha(p_1, p_2, p_3) + \alpha(q_1, q_2, q_3) \\
&= \alpha\mathbf{P} + \alpha\mathbf{Q},
\end{aligned}
$$

as was to be shown. ■

The Representation of Vectors in Terms of the Unit Vectors i, j, and k

The components of a vector, together with vector addition, can be used to describe vectors in a very convenient way. The idea is simple, and it involves using the standard convention that **i**, **j**, and **k** are vectors of unit length that point in the positive sense along the x-, y-, and z-axes, respectively. Vectors such as **i**, **j**, and **k** that have a unit norm (length) are called **unit vectors**, so $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$.

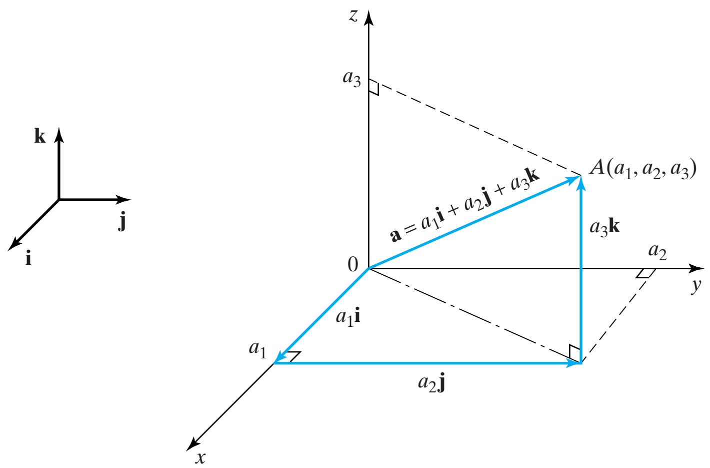


FIGURE 2.9 Vector **a** in terms of the unit vectors **i**, **j**, and **k**.

An arbitrary vector **a** can be represented by an "arrow," with its base at the origin and its tip at the point A with cartesian coordinates (a_1, a_2, a_3) where, of course, a_1, a_2, and a_3 are also the components of **a**. Consequently, scaling the unit vectors **i**, **j**, and **k** by the respective x, y, and z components a_1, a_2, and a_3 of **a**, followed by vector addition of these three vectors, shows that **a** can be written

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \tag{1}$$

as can be seen from Fig. 2.9. The representation of vector **a** in terms of the unit vectors **i**, **j**, and **k** in (1), and the ordered triple notation, are equivalent, so

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3). \tag{2}$$

position vector

In some applications a vector defines a point in space, so vectors of this type are called **position vectors**. The symbol **r** is normally used for a position vector, so if point P with coordinates (x, y, z) is a general point in space, as in Fig. 2.10, its

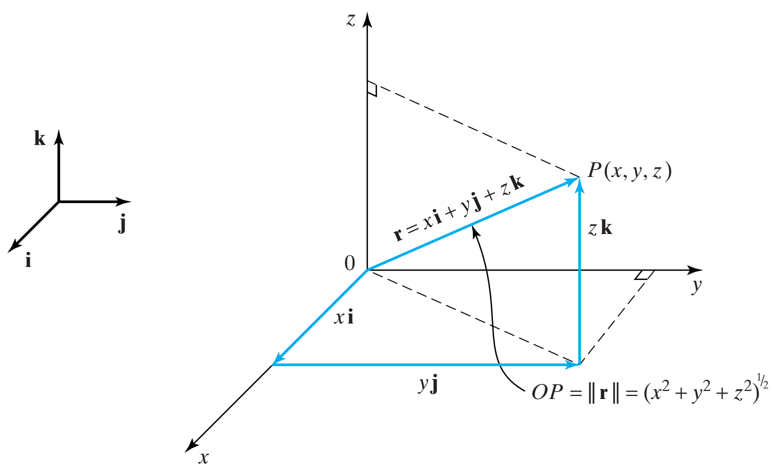


FIGURE 2.10 Position vector of a general point P in space.

position vector relative to the origin is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \tag{3}$$

and its norm (length) is

$$\|\mathbf{r}\| = (x^2 + y^2 + z^2)^{1/2}. \tag{4}$$

EXAMPLE 2.6 (a) Find the distance of point P from the origin given that its position vector is $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$. (b) If a general point P in space has position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, describe the surface defined by $\|\mathbf{r}\| = 3$ and find its cartesian equation.

Solution **(a)** As $\mathbf{r}$ is the position vector of P relative to the origin, the distance of point P from the origin is $\|\mathbf{r}\| = [2^2 + 4^2 + (-3)^2]^{1/2} = \sqrt{29}$.
(b) As $\|\mathbf{r}\| = 3$ (constant), it follows that the required surface is one for which every point lies at a distance 3 from the origin, so the surface must be a sphere of radius 3 centered on the origin. As $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the general position vector of a point on this sphere, the result $\|\mathbf{r}\| = 3$ is equivalent to $(x^2 + y^2 + z^2)^{1/2} = 3$, so the cartesian equation of the sphere is $x^2 + y^2 + z^2 = 9$. ■

Because of the equivalence of the ordered number triple notation and the representation of vectors in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ given in (2), both systems obey the same rules governing the addition and scaling of vectors in terms of their components. Thus, the following rules apply to the combination of any two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ expressed in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and an arbitrary real number λ.

The sum $\mathbf{a} + \mathbf{b}$ is given by

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}. \tag{5}$$

The product $\lambda\,\mathbf{a}$ is given by

$$\lambda\,\mathbf{a} = \lambda a_1\,\mathbf{i} + \lambda a_2\,\mathbf{j} + \lambda a_3\,\mathbf{k}. \tag{6}$$

The norm of scaled vector $\lambda\,\mathbf{a}$ is given by

$$\begin{aligned}\|\lambda\,\mathbf{a}\| &= |\lambda|\cdot\|\mathbf{a}\| \\ &= |\lambda|\left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}.\end{aligned} \tag{7}$$

EXAMPLE 2.7 If $\mathbf{a} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$, find (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} - \mathbf{b}$, (c) $2\mathbf{a} + \mathbf{b}$, and (d) $|-2\mathbf{a}|$.

Solution

(a)
$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (5\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + (2\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) \\ &= (5 + 2)\mathbf{i} + (1 - 2)\mathbf{j} + (-3 - 7)\mathbf{k} \\ &= 7\mathbf{i} - \mathbf{j} - 10\mathbf{k}.\end{aligned}$$

(b)
$$\begin{aligned}\mathbf{a} - \mathbf{b} &= (5\mathbf{i} + \mathbf{j} - 3\mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}) \\ &= (5 - 2)\mathbf{i} + (1 - (-2))\mathbf{j} + (-3 - (-7))\mathbf{k} \\ &= 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.\end{aligned}$$

(c)
$$\begin{aligned}2\mathbf{a}+\mathbf{b} &= 2(5\mathbf{i}+\mathbf{j}-3\mathbf{k})+(2\mathbf{i}-2\mathbf{j}-7\mathbf{k})\\ &= (10\mathbf{i}+2\mathbf{j}-6\mathbf{k})+(2\mathbf{i}-2\mathbf{j}-7\mathbf{k})\\ &= (10+2)\mathbf{i}+(2+(-2))\mathbf{j}+(-6+(-7))\mathbf{k}\\ &= 12\mathbf{i}-13\mathbf{k}.\end{aligned}$$

(d)
$$|-2\mathbf{a}| = [(-10)^2+(-2)^2+6^2]^{1/2} = 2\sqrt{35}$$

or, equivalently,

$$|-2\mathbf{a}| = |-2|\cdot\|\mathbf{a}\| = 2\|\mathbf{a}\| = 2[5^2+1^2+(-3)^2]^{1/2} = 2\sqrt{35}. \quad ■$$

Finding a Unit Vector in the Direction of an Arbitrary Vector

It is often necessary to find a unit vector in the direction of an arbitrary vector $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. This is accomplished by dividing $\mathbf{a}$ by its norm $\|\mathbf{a}\|$, because the vector $\mathbf{a}/\|\mathbf{a}\|$ has the same sense as $\mathbf{a}$ and its norm is 1. It is convenient to use a symbol related to an arbitrary vector $\mathbf{a}$ to indicate the unit vector in its direction, so from now on such a vector will be denoted by $\hat{\mathbf{a}}$, read "**a** hat." So if $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$,

$$\begin{aligned}\hat{\mathbf{a}} &= \mathbf{a}/\|\mathbf{a}\| = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})/\left(a_1^2+a_2^2+a_3^2\right)^{1/2}\\ &= (a_1/a)\mathbf{i}+(a_2/a)\mathbf{j}+(a_3/a)\mathbf{k}, \quad \text{with } a = \left(a_1^2+a_2^2+a_3^2\right)^{1/2}.\end{aligned} \tag{8}$$

As the symbols $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are used exclusively for the unit vectors in the x-, y-, and z-directions, it is not necessary to write $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.

The relationship between $\mathbf{a}$, $\hat{\mathbf{a}}$, and $\|\mathbf{a}\|$ can be put in the useful form

$$\mathbf{a} = \|\mathbf{a}\|\hat{\mathbf{a}}, \tag{9}$$

showing that a general vector $\mathbf{a}$ can always be written as the unit vector $\hat{\mathbf{a}}$ scaled by $\|\mathbf{a}\|$. Unless otherwise stated, $\mathbf{a} \neq \mathbf{0}$.

EXAMPLE 2.8 Find a unit vector in the direction of $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+5\mathbf{k}$.

Solution As $\|\mathbf{a}\| = (3^2+2^2+5^2)^{1/2} = \sqrt{38}$, it follows that

$$\hat{\mathbf{a}} = \mathbf{a}/\|\mathbf{a}\| = (3/\sqrt{38})\mathbf{i}+(2/\sqrt{38})\mathbf{j}+(5/\sqrt{38})\mathbf{k}. \quad ■$$

EXAMPLE 2.9 It is known from experiments in mechanics that forces are vector quantities and so combine according to the laws of vector algebra. Use this fact to find the sum and difference of a force of 9 units in the direction of $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and a force of 10 units in the direction of $4\mathbf{i}-3\mathbf{j}$, and determine the magnitudes of these forces.

Solution We will use the convention that a unit vector represents a force of 1 unit. Let $\mathbf{F}$ be the force of 9 units. Then as $\|2\mathbf{i}+\mathbf{j}-2\mathbf{k}\| = [2^2+1^2+(-2)^2]^{1/2} = 3$, the unit vector in the direction of $\mathbf{F}$ is

$$\hat{\mathbf{F}} = (1/3)(2\mathbf{i}+\mathbf{j}-2\mathbf{k}) = (2/3)\mathbf{i}+(1/3)\mathbf{j}-(2/3)\mathbf{k},$$

so $\mathbf{F} = 9\hat{\mathbf{F}} = 6\mathbf{i}+3\mathbf{j}-6\mathbf{k}$ units.

Similarly, let $\mathbf{G}$ be the force of 10 units. Then as $\|4\mathbf{i} - 3\mathbf{j}\| = 5$, the unit vector in the direction of $\mathbf{G}$ is

$$\hat{\mathbf{G}} = (1/5)(4\mathbf{i} - 3\mathbf{j}) = (4/5)\mathbf{i} - (3/5)\mathbf{j},$$

so $\mathbf{G} = 10\hat{\mathbf{G}} = 8\mathbf{i} - 6\mathbf{j}$ units.

Combining these results shows that $\mathbf{F} + \mathbf{G} = 14\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$ units, and $\mathbf{F} - \mathbf{G} = -2\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$ units, from which it follows that the magnitudes of the forces are given by

$$\|\mathbf{F} + \mathbf{G}\| = \sqrt{241} \text{ units and } \|\mathbf{F} - \mathbf{G}\| = 11 \text{ units.} \qquad \blacksquare$$

Equality of vectors expressed in terms of unit vectors

As the difference of two equal and opposite vectors is the null vector $\mathbf{0}$, this shows that if $\mathbf{a} = \mathbf{b}$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then the respective components of vectors $\mathbf{a}$ and $\mathbf{b}$ must be equal, leading to the result that

$$\mathbf{a} = \mathbf{b} \text{ if, and only if, } a_1 = b_1,\ a_2 = b_2, \text{ and } a_3 = b_3. \tag{10}$$

Simple Geometrical Applications of Vectors

Although our use of vectors will be mainly in connection with the calculus, the following simple geometrical applications are helpful because they illustrate basic vector arguments and properties.

Although we have seen how an arbitrary vector can be expressed in terms of unit vectors associated with a cartesian coordinate system, it must be remembered that the fundamental concept of a vector and its algebra is independent of a coordinate system. Because of this, it is often possible to use the rules governing elementary vector algebra given in Theorem 2.1 to establish equations in a purely vectorial manner, without the need to appeal to any coordinate system. Once a general vector equation has been established, the representation of the vectors involved in terms of their components and the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ can be used to convert the vector equation into the equivalent cartesian equations.

The purely vectorial approach to geometrical problems is well illustrated by finding the vector $\underline{AB}$ in terms of the position vectors of points A and B, and then using the result to find the position vector of the mid-point of $\underline{AB}$. After this, the purely vectorial derivation of a geometrical result followed by its interpretation in cartesian form will be illustrated by finding the equation of a straight line in three space dimensions.

Vector $\underline{AB}$ in terms of the position vectors of A and B

Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of points A and B relative to an origin O, as shown in Fig. 2.11.

An application of the triangle rule for the addition of vectors gives

$$\underline{OA} + \underline{AB} = \underline{OB},$$

but $\underline{OA} = \mathbf{a}$ and $\underline{OB} = \mathbf{b}$, so

$$\mathbf{a} + \underline{AB} = \mathbf{b},$$

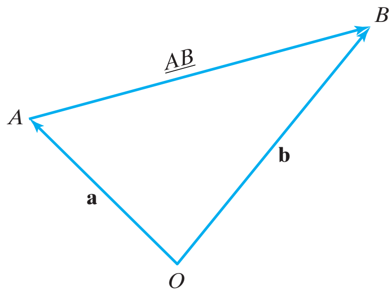


FIGURE 2.11 Vectors **a**, **b**, and $\underline{AB}$.

giving

$$\underline{AB} = \mathbf{b} - \mathbf{a}. \tag{11}$$

When expressed in words, this simple but useful result asserts that vector $\underline{AB}$ is obtained by subtracting the position vector **a** of point A from the position vector **b** of point B.

EXAMPLE 2.10

Find the position vector of the mid-point of $\underline{AB}$ if point A has position vector **a** and point B has position vector **b** relative to an origin O.

Solution Let point C, with position vector **c** relative to origin O, be the mid-point of $\underline{AB}$, as shown in Fig. 2.12.
By the triangle rule,

$$\underline{OA} + \underline{AC} = \underline{OC},$$

but $\underline{OA} = \mathbf{a}$, and from (11) $\underline{AC} = (1/2)(\mathbf{b} - \mathbf{a})$, so

$$\underline{OC} = \mathbf{a} + (1/2)(\mathbf{b} - \mathbf{a}),$$

so the required result is

$$\mathbf{c} = \underline{OC} = (1/2)(\mathbf{b} + \mathbf{a}).$$

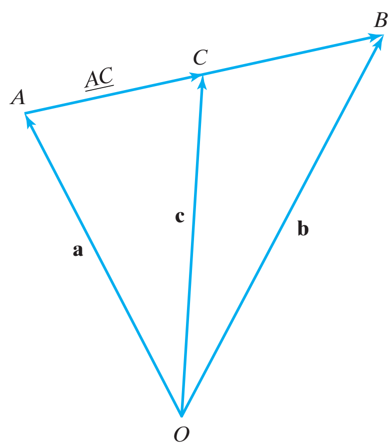


FIGURE 2.12 C is the mid-point of $\underline{AB}$.

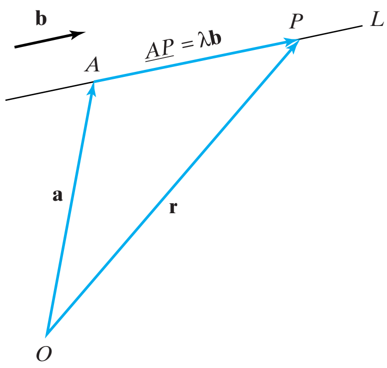


FIGURE 2.13 The straight line L.

The vector and cartesian equations of a straight line

Let line L be a straight line through point A with position vector $\mathbf{a}$ relative to an origin O, and let the line be parallel to a vector $\mathbf{b}$. If P is an arbitrary point on line L with position vector $\mathbf{r}$ relative to O, an application of the triangle rule for vector addition to the vectors shown in Fig. 2.13 gives

$$\mathbf{r} = \underline{OA} + \underline{AP}.$$

vector equation of straight line

But $\underline{OA} = \mathbf{a}$, and as $\underline{AP}$ is parallel to $\mathbf{b}$, a number λ can always be found such that $\underline{AP} = \lambda\mathbf{b}$, so the **vector equation** of line L becomes

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}. \tag{12}$$

Notice that result (12) determines all points P on L if λ is taken to be a number in the interval $-\infty < \lambda < \infty$.

The cartesian equations of line L follow by setting $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in result (12), and then using the definition of equality of vectors given in (10) to obtain the corresponding three scalar cartesian equations. Proceeding in this way we find that

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + \lambda(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}),$$

cartesian and standard form of straight line

so equating corresponding components of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ on each side of this equation brings us to the required **cartesian equations** for L in the form

$$x_1 = a_1 + \lambda b_1, \quad x_2 = a_2 + \lambda b_2, \quad x_3 = a_3 + \lambda b_3. \tag{13}$$

An equivalent form of these equations is obtained by solving each equation for λ and equating the results to get

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \lambda. \tag{14}$$

This is the **standard form** (also called the **canonical form**) of the cartesian equations of a straight line. It is important to notice that when written in standard form the coefficients of x, y, and z are all *unity*. Once the equation of a straight line is written in standard form, equating each numerator to zero determines the components (a_1, a_2, a_3) of a position vector of a point on the line, while the denominators in the order (b_1, b_2, b_3) determine the components of a vector parallel to the line.

EXAMPLE 2.11 A straight line L is given in the form

$$\frac{2x-3}{4} = \frac{3-y}{2} = \frac{z+1}{3}.$$

Find the position vector of a point on L and a vector parallel to L.

Solution When the equation is written in standard form it becomes

$$\frac{x-3/2}{2} = \frac{y-3}{-2} = \frac{z+1}{3} = \lambda.$$

Comparing these equations with (14) shows that $(a_1, a_2, a_3) = (3/2, 3, -1)$ and $\mathbf{b} = (b_1, b_2, b_3) = (2, -2, 3)$. So the position vector of a point on the line is $\mathbf{a} = (3/2)\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, and a vector parallel to the line is $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Neither of these results is unique, because $\mu\mathbf{b}$ is also parallel to the line for any scalar $\mu \neq 0$, and any other point on L would suffice. For example, the vector $14\mathbf{i} - 14\mathbf{j} + 21\mathbf{k}$ is also parallel to the line, while setting $\lambda = 2$ leads to the result $(a_1, a_2, a_3) = (11/2, -1, 5)$, corresponding to a different point on the same line, this time with position vector $\mathbf{a} = (11/2)\mathbf{i} - \mathbf{j} + 5\mathbf{k}$. ■

Summary

This section has introduced vectors both as geometrical quantities that can be represented by directed line segments and, using a right-handed system of cartesian axes, as ordered number triples. Definitions of the scaling, addition, and subtraction of vectors have been given, and a general vector has been defined in terms of the set of three unit vectors **i**, **j**, and **k** that lie along the orthogonal cartesian axes O{x, y, z}. Finally, the vector and cartesian equations of a straight line in space have been derived, and the standard form of the cartesian equations has been introduced from which a vector parallel to the line may be found by inspection.

EXERCISES 2.1

1. Prove Results 1, 3, and 6 of Theorem 2.1.
2. Given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, find (a) $\mathbf{a} + 2\mathbf{b} - \mathbf{c}$, (b) a vector $\mathbf{d}$ such that $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$, and (c) a vector $\mathbf{d}$ such that $\mathbf{a} - \mathbf{b} + \mathbf{c} + 3\mathbf{d} = \mathbf{0}$.
3. Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, find (a) a vector $\mathbf{c}$ such that $2\mathbf{a} + \mathbf{b} + 2\mathbf{c} = \mathbf{i} + \mathbf{k}$, (b) a vector $\mathbf{c}$ such that $3\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
4. Given that $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$, find (a) $2\mathbf{a} + 3\mathbf{b} - 3\mathbf{c}$, (b) a vector $\mathbf{d}$ such that $\mathbf{a} + 3\mathbf{b} - 2\mathbf{c} + 3\mathbf{d} = \mathbf{0}$, and (c) a vector $\mathbf{d}$ such that $2\mathbf{a} - 3\mathbf{d} = \mathbf{b} + 4\mathbf{c}$.
5. Given that A and B have the respective position vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, find the vector $\underline{AB}$ and a unit vector in the direction of $\underline{AB}$.
6. Given that A and B have the respective position vectors $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, find the vector $\underline{AB}$ and the position vector $\mathbf{c}$ of the mid-point of $\underline{AB}$.
7. Given that A and B have the respective position vectors $\mathbf{a}$ and $\mathbf{b}$, find the position vector of a point P on the line AB located between A and B such that

$$(\text{length } AP)/(\text{length } PB) = m/n, \quad \text{where } m, n > 0$$

are any two real numbers.

8. Find the position vector $\mathbf{r}$ of a point P on the straight line joining point A at $(1, 2, 1)$ and point B at $(3, -1, 2)$ and between A and B such that

$$(\text{length } AP)/(\text{length } PB) = 3/2.$$

9. It is known from Euclidean geometry that the medians of a triangle (lines drawn from a vertex to the mid-point of the opposite side) all meet at a single point P, and that P is two-thirds of the distance along each median from the vertex through which it passes. If the vertices A, B, and C of a triangle have the respective position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, show that the position vector of P is $(1/3)(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

10. Forces of 1, 2, and 3 units act through the origin along, and in the positive directions of, the respective x-, y-, and z-axes. Find the vector sum $\mathbf{S}$ of these forces, the magnitude $\|\mathbf{S}\|$ of the sum of the vectors, and a unit vector in the direction of $\mathbf{S}$.

11. Forces of 2, 1, and 4 units act through the origin along, and in the positive directions of, the respective x-, y-, and z-axes. Find the vector sum $\mathbf{S}$ of these forces, the magnitude $\|\mathbf{S}\|$ of the sum of the vectors, and a unit vector in the direction of $\mathbf{S}$.

12. A straight line L is given in the form

$$\frac{3x-1}{4} = \frac{2y+3}{2} = \frac{2-3z}{1}.$$

Find the position vectors of two different points on L and a unit vector parallel to L.

13. A straight line L is given in the form

$$\frac{2x+1}{3} = \frac{3y+2}{4} = \frac{2-4z}{-1}.$$

Find position vectors of two different points on L and a unit vector parallel to L.

14. Given that a straight line L_1 passes through the points $(-2, 3, 1)$ and $(1, 4, 6)$, find (a) the position vector of a point on the line and a vector parallel to it, and (b) a straight line L_2 parallel to L_1 that passes through the point $(1, 2, 1)$.

15. Given that a straight line L_1 passes through the points $(3, 2, 4)$ and $(2, 1, 6)$, find (a) the position vector of a point on the line and a vector parallel to it, and (b) a straight line L_2 parallel to L_1 that passes through the point $(-2, 1, 2)$.

16. A straight line has the vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a} = 3\mathbf{j} + 2\mathbf{k}$, and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Find the cartesian equations of the line and the coordinates of three points that lie on it.

17. A straight line passes through the point $(3, 2, -3)$ parallel to the vector $2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Find the cartesian equations of the line and the coordinates of three points that lie on it.

18. In mechanics, if a point A moves with velocity $\mathbf{v}_A$ and point B moves with velocity $\mathbf{v}_B$, the velocity $\mathbf{v}_R$ of A **relative** to B (the **relative velocity** of A with respect to B) is defined as $\mathbf{v}_R = \mathbf{v}_A - \mathbf{v}_B$. Power boat A moves northeast at 20 knots and power boat B moves southeast at 30 knots. Find the velocity of boat A relative to boat B, and a unit vector in the direction of the relative velocity.

2.2 The Dot Product (Scalar Product)

A product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can be formed in such a way that the result is a scalar. The result is written $\mathbf{a} \cdot \mathbf{b}$ and called the **dot product** of $\mathbf{a}$ and $\mathbf{b}$. The names **scalar product** and **inner product** are also used in place of the term *dot product*.

Dot Product

dot or scalar product

Let $\mathbf{a}$ and $\mathbf{b}$ be any two vectors that after a translation to bring their bases into coincidence are inclined to one another at an angle θ, as shown in Fig. 2.14, where $0 \leq \theta \leq \pi$. Then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is defined as the number

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\theta.$$

This geometrical definition of the dot product has many uses, but when working with vectors $\mathbf{a}$ and $\mathbf{b}$ that are expressed in terms of their components in the $\mathbf{i}$, $\mathbf{j}$, and

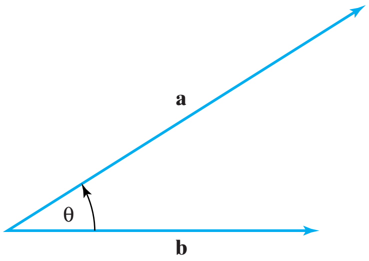


FIGURE 2.14 Vectors **a** and **b** inclined at an angle θ.

k directions, a more convenient form is needed. An equivalent definition that is easier to use is given later in (23).

properties of the dot product

Properties of the dot product

The following results, in which **a** and **b** are any two vectors and λ and μ are any two scalars, are all immediate consequences of the definition of the dot product.

The dot product is commutative

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a} \quad \text{and} \quad \lambda\mathbf{a}\cdot\mu\mathbf{b}=\mu\mathbf{a}\cdot\lambda\mathbf{b}=\lambda\mu\mathbf{a}\cdot\mathbf{b} \tag{15}$$

The dot product is distributive and linear

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} \quad \text{and} \quad \mathbf{a}\cdot(\lambda\mathbf{b}+\mu\mathbf{c})=\lambda\mathbf{a}\cdot\mathbf{b}+\mu\mathbf{a}\cdot\mathbf{c}. \tag{16}$$

The angle between two vectors

The angle θ between vectors **a** and **b** is given by

$$\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\cdot\|\mathbf{b}\|}, \quad \text{with } 0\le\theta\le\pi. \tag{17}$$

Parallel vectors ($\theta = 0$)

If vectors **a** and **b** are parallel, then

$$\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\cdot\|\mathbf{b}\| \quad \text{and, in particular,} \quad \mathbf{a}\cdot\mathbf{a}=\|\mathbf{a}\|^2. \tag{18}$$

Orthogonal vectors ($\theta = \pi/2$)

If vectors **a** and **b** are orthogonal, then

$$\mathbf{a}\cdot\mathbf{b}=0. \tag{19}$$

Product of unit vectors

If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors, then

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos\theta, \quad \text{with } 0 \le \theta \le \pi. \tag{20}$$

An immediate consequence of properties (15), (19), and (20) is that

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \tag{21}$$

and

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0. \tag{22}$$

We now use results (21) and (22) to arrive at a simple expression for the dot product in terms of the components of $\mathbf{a}$ and $\mathbf{b}$. To arrive at the result we set $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and form the dot product

$$\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}).$$

dot product in terms of components

Expanding this product using (15) and (16) and making use of results (21) and (22) brings us to the following *alternative definition* of the **dot product** expressed in terms of the components of $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3. \tag{23}$$

Using (23) in (17) produces the following useful expression that can be used to find the angle θ between $\mathbf{a}$ and $\mathbf{b}$:

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}\left(b_1^2 + b_2^2 + b_3^2\right)^{1/2}} \text{ where } 0 \le \theta \le \pi. \tag{24}$$

EXAMPLE 2.12

Find $\mathbf{a}\cdot\mathbf{b}$ and the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, given that $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Solution $\|\mathbf{a}\| = \sqrt{14}$, $\|\mathbf{b}\| = 3$, and $\mathbf{a}\cdot\mathbf{b} = 1\cdot 2 + 2\cdot(-1) + 3\cdot(-2) = -6$. Using these results in (24) gives

$$\cos\theta = -6/(3\sqrt{14}) = -2/\sqrt{14},$$

so as $0 \le \theta \le \pi$ we see that $\theta = 2.1347$ radians, or $\theta = 122.3°$. ■

projecting a vector onto a line

The projection of a vector onto the line of another vector

The projection of vector $\mathbf{a}$ onto the line of vector $\mathbf{b}$ is a scalar, and it is the *signed* length of the geometrical projection of vector $\mathbf{a}$ onto a line parallel to $\mathbf{b}$, with the sign positive for $0 \le \theta < \pi/2$ and negative for $\pi/2 < \theta \le \pi$. This is illustrated in Fig. 2.15, from which it is seen that the signed length of the projection of $\mathbf{a}$ onto the line of vector $\mathbf{b}$ is ON, where $ON = \|\mathbf{a}\|\cos\theta$.

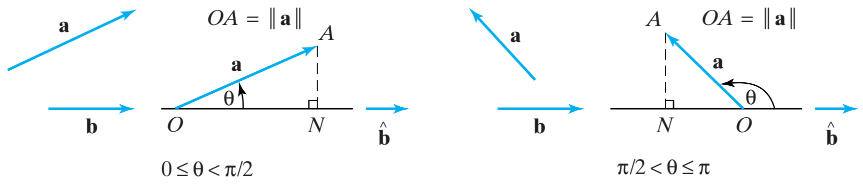


FIGURE 2.15 The projection of vector **a** onto the line of vector **b**.

If $\hat{\mathbf{b}}$ is the unit vector along $\mathbf{b}$, then as $\mathbf{a} = \hat{\mathbf{a}}\|\mathbf{a}\|$, and $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos\theta$, the projection $ON = \|\mathbf{a}\|\cos\theta$ can be written as the dot product

$$ON = \|\mathbf{a}\|\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \mathbf{a} \cdot \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \tag{25}$$

EXAMPLE 2.13 Find the strength of the magnetic field vector $\mathbf{H} = 5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ in the direction of $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, where a unit vector represents one unit of magnetic flux.

Solution We are required to find the projection of vector $\mathbf{H}$ in the direction of the vector $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Setting $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\|\mathbf{b}\| = 3$, so $\hat{\mathbf{b}} = (1/3)(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$, so the strength of the vector $\mathbf{H}$ in the direction of $\mathbf{b}$ is

$$\mathbf{H} \cdot \hat{\mathbf{b}} = (1/3)(5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 7.$$

■

Direction cosines and direction ratios

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is an arbitrary vector, the unit vector $\hat{\mathbf{a}}$ in the direction of $\mathbf{a}$ is

$$\begin{aligned}\hat{\mathbf{a}} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})/\|\mathbf{a}\| \\ &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})/\left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}.\end{aligned} \tag{26}$$

Taking the dot product of $\mathbf{a}$ with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and setting $l = a_1/(a_1^2 + a_2^2 + a_3^2)^{1/2}$, $m = a_2/(a_1^2 + a_2^2 + a_3^2)^{1/2}$, and $n = a_3/(a_1^2 + a_2^2 + a_3^2)^{1/2}$ gives

$$l = \mathbf{i} \cdot \hat{\mathbf{a}}, \quad m = \mathbf{j} \cdot \hat{\mathbf{a}}, \quad \text{and} \quad n = \mathbf{k} \cdot \hat{\mathbf{a}},$$

so we may write

$$\hat{\mathbf{a}} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}. \tag{27}$$

The dot product $\hat{\mathbf{a}} \cdot \hat{\mathbf{a}} = l^2 + m^2 + n^2 = (a_1^2 + a_2^2 + a_3^2)/\|\mathbf{a}\|^2$, but $\|\mathbf{a}\|^2 = a_1^2 + a_2^2 + a_3^2$, so

$$l^2 + m^2 + n^2 = 1. \tag{28}$$

The number l is the cosine of the angle β_1 between $\mathbf{a}$ and the x-axis, the number m is the cosine of the angle β_2 between $\mathbf{a}$ and the y-axis, and the number n is the cosine of the angle β_3 between $\mathbf{a}$ and the z-axis, as shown in

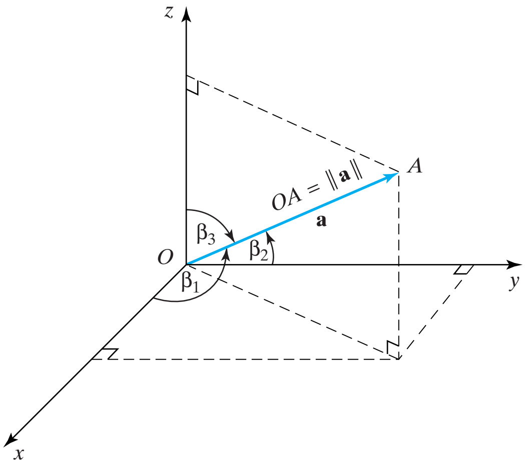


FIGURE 2.16 The angles β_1, β_2, and β_3.

direction cosines

Fig. 2.16. The numbers (l, m, n) are called the **direction cosines** of **a**, because they determine the direction of the unit vector $\hat{\mathbf{a}}$ that is parallel to **a**.

Notice that when any two of the three direction cosines l, m, and n of a vector **a** are given, the third is related to them by

$$l^2 + m^2 + n^2 = 1.$$

Because of result (27) it is always possible to write

$$\mathbf{a} = \|\mathbf{a}\|(l\mathbf{i} + m\mathbf{j} + n\mathbf{k}), \tag{29}$$

where l, m, and n are the direction cosines of **a**.

direction ratios

As the components a_1, a_2, and a_3 of **a** are *proportional* to the direction cosines, they are called the **direction ratios** of **a**.

EXAMPLE 2.14

Find the direction cosines and direction ratios of $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution As $\|\mathbf{a}\| = \sqrt{14}$, the direction cosines are $l = 3/\sqrt{14}$, $m = 1/\sqrt{14}$, and $n = -2/\sqrt{14}$. The direction ratios of **a** are 3, 1, and -2, or any nonnegative multiple of these three numbers such as $15/\sqrt{14}$, $5/\sqrt{14}$, and $-10/\sqrt{14}$. ■

The triangle inequality

The following result will be needed in the proof of the triangle inequality that is to follow. The absolute value of $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\cdot\|\mathbf{b}\|\cos\theta$ is

$$|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\cdot\|\mathbf{b}\||\cos\theta|,$$

but $|\cos\theta| \le 1$, so using this in the above result we obtain the **Cauchy–Schwarz inequality**,

$$|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\cdot\|\mathbf{b}\|. \tag{30}$$

THEOREM 2.2

The triangle inequality If $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|.$$

Proof From (18) we have

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}\\ &= \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2,\end{aligned}$$

but $\mathbf{a}\cdot\mathbf{b} \le |\mathbf{a}\cdot\mathbf{b}|$, so from the Cauchy–Schwarz inequality (30)

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &\le \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\cdot\|\mathbf{b}\| + \|\mathbf{b}\|^2\\ &= (\|\mathbf{a}\| + \|\mathbf{b}\|)^2.\end{aligned}$$

Taking the positive square root of this last result, we obtain the triangle inequality

$$\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|. \qquad \blacksquare$$

The triangle inequality will be generalized in Section 2.5, but in its present form it is the vector equivalent of the Euclidean theorem that "the sum of the lengths of any two sides of a triangle is greater than or equal to the length of the third side," and it is from this theorem that the inequality derives its name.

Equation of a Plane

When working with the vector calculus it is sometimes necessary to consider a plane that is locally tangent to a point on a surface in space so it will be useful to derive the general equation of a plane in both its vector and cartesian forms.

A plane Π can be defined by specifying a fixed point belonging to the plane and a vector $\mathbf{n}$ that is perpendicular to the plane. This follows because if $\mathbf{n}$ is perpendicular at a point on the plane, it must be perpendicular at every point on the plane. Any vector $\mathbf{n}$ that is perpendicular to a plane is called a **normal** to the plane. Clearly a normal to a plane is not unique, because a plane has two sides, so if a normal $\mathbf{n}$ is directed away from one side of the plane, the vector $-\mathbf{n}$ is a normal directed away from the other side. Both $\mathbf{n}$ and $-\mathbf{n}$ can be scaled by any nonzero number and still remain normals; consequently, if $\mathbf{n}$ is a normal to a plane, so also are all vectors of the form $\lambda\mathbf{n}$, with $\lambda \ne 0$ any real number.

Let a fixed point A on plane Π with normal $\mathbf{n}$ have position vector $\mathbf{a}$ relative to an origin O, and let P be a general point on plane Π with position vector $\mathbf{r}$ relative to O. Then, as may be seen from Fig. 2.17, the vector $\mathbf{r}-\mathbf{a}$ lies in the plane, and so is perpendicular (normal) to $\mathbf{n}$. Forming the dot product of $\mathbf{n}$ and $\mathbf{r}-\mathbf{a}$, and using (19), shows that the **vector equation** of plane Π is

vector equation of a plane

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{a}) = 0, \tag{31}$$

or, equivalently,

$$\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{a}. \tag{32}$$

cartesian equation of a plane

The **cartesian form** of this equation follows by considering a general point with coordinates (x, y, z) on plane Π, setting $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$,

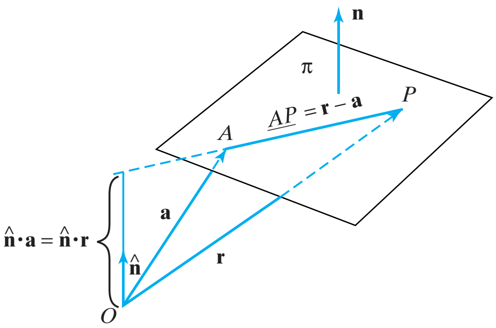


FIGURE 2.17 Plane Π with normal **n** passing through point A.

and $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$, and then substituting into (32) to get

$$(n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}) \cdot (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).$$

Taking the dot products and using results (21) and (22) show the cartesian equation of plane Π to be

$$n_1x + n_2y + n_3z = n_1a_1 + n_2a_2 + n_3a_3 = d, \text{ a constant.} \tag{33}$$

EXAMPLE 2.15 Find the cartesian equation of the plane through the point $(2, 5, 3)$ with normal $3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$.

Solution Here $n_1 = 3$, $n_2 = 2$, $n_3 = -7$ and $a_1 = 2$, $a_2 = 5$, and $a_3 = 3$, so substituting into (33) shows the plane has the equation

$$3x + 2y - 7z = -5.$$ ■

Summary

This section has introduced the dot or scalar product of two vectors in geometrical terms and, more conveniently for calculations, in terms of the components of the two vectors involved. The applications given include the important operation of projecting a vector onto the line of another vector and the derivation of the vector equation and cartesian equation of a plane.

EXERCISES 2.2

1. Find the dot products of the following pairs of vectors:
   (a) $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.
   (b) $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
   (c) $\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + \mathbf{k}$.
2. Find the dot products of the following pairs of vectors:
   (a) $\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
   (b) $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.
   (c) $5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$.
3. Find which of the following pairs of vectors are orthogonal:
   (a) $3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$, $-9\mathbf{i} - 6\mathbf{j} + 18\mathbf{k}$.
   (b) $3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$, $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
   (c) $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + \mathbf{j} - \mathbf{k}$.
   (d) $\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + \mathbf{k}$.
4. Find which, if any, of the following pairs of vectors are orthogonal:
   (a) $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $8\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
   (b) $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$.
   (c) $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
   (d) $\mathbf{i} + \mathbf{j}$, $2\mathbf{j} + 3\mathbf{k}$.
5. Given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$, find (a) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$. (b) $(2\mathbf{b} - 3\mathbf{c}) \cdot \mathbf{a}$. (c) $\mathbf{a} \cdot \mathbf{a}$. (d) $\mathbf{c} \cdot (\mathbf{a} - 2\mathbf{b})$.

6. Given that $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and $\mathbf{c} = 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, find (a) $\mathbf{b} \cdot (\mathbf{b} + (\mathbf{a} \cdot \mathbf{c})\mathbf{c})$. (b) $(\mathbf{a} + 2\mathbf{b}) \cdot (2\mathbf{b} - 3\mathbf{c})$. (c) $(\mathbf{c} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{a})\mathbf{c}$.

7. Find the angle between the following pairs of vectors:
(a) $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.
(b) $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
(c) $3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.
(d) $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $4\mathbf{i} - 8\mathbf{j} + 16\mathbf{k}$.

8. Given $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, and $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, find the angles between the following pairs of vectors:
(a) $\mathbf{a} + \mathbf{b}$, $\mathbf{b} - 2\mathbf{c}$. (b) $2\mathbf{a} - \mathbf{c}$, $\mathbf{a} + \mathbf{b} - \mathbf{c}$. (c) $\mathbf{b} + 3\mathbf{c}$, $\mathbf{a} - 2\mathbf{c}$.

9. Find the component of the force $\mathbf{F} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ in the direction of the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

10. Find the component of the force $\mathbf{F} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ in the direction of the vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

11. Given that $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, find (a) the projection of $\mathbf{a}$ onto the line of $\mathbf{b}$, and (b) the projection of $\mathbf{b}$ onto the line of $\mathbf{a}$.

12. Given that $\mathbf{a} = 3\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, (a) find the projection of $\mathbf{a}$ onto the line of $\mathbf{b}$ and (b) compare the magnitude of $\mathbf{a}$ with the result found in (a) and comment on the result.

13. Find the direction cosines and corresponding angles for the following vectors:
(a) $\mathbf{i} + \mathbf{j} + \mathbf{k}$. (b) $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. (c) $4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

14. Find the direction cosines and corresponding angles for the following vectors:
(a) $\mathbf{i} - \mathbf{j} - \mathbf{k}$. (b) $2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$. (c) $-4\mathbf{j} - \mathbf{k}$.

15. Verify the triangle inequality for vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 7\mathbf{k}$.

16. Verify the triangle inequality for vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

17. Find the equation of the plane with normal $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ that contains the point $(1, 0, 1)$.

18. Find the equation of the plane with normal $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ that contains the point $(2, -3, 4)$.

19. Given that a plane passes through the point $(2, 3, -5)$, and the vector $2\mathbf{i} + \mathbf{k}$ is normal to the plane, find the cartesian form of its equation.

20. The equation of a plane is $3x + 2y - 5z = 4$. Find a vector that is normal to the plane, and the position vector of a point on the plane.

21. Explain why if the vector equation of plane Π in (32) is divided by $\|\mathbf{n}\|$ to bring it into the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, the number $|\mathbf{a} \cdot \mathbf{n}|$ is the perpendicular distance of origin O from the plane. Explain also why if $\mathbf{a} \cdot \mathbf{n} > 0$ the plane lies to the side of O toward which $\mathbf{n}$ is directed, as in Fig. 2.15, but that if $\mathbf{a} \cdot \mathbf{n} < 0$ it lies on the opposite side of O toward which $-\mathbf{n}$ is directed.

22. Use the result of Exercise 21 to find the perpendicular distance of the plane $2x - 4y - 5z = 5$ from the origin.

23. The angle between two planes is defined as the angle between their normals. Find the angle between the two planes $x + 3y + 2z = 4$ and $2x - 5y + z = 2$.

24. Find the angle between the two planes $3x + 2y - 2z = 4$ and $2x + y + 2z = 1$.

25. Let $\mathbf{a}$ and $\mathbf{b}$ be two arbitrary skew (nonparallel) vectors, and set $\mathbf{a} = \mathbf{a}_b + \mathbf{a}_p$, where $\mathbf{a}_b$ is parallel to $\mathbf{b}$ and $\mathbf{a}_p$ is perpendicular to $\mathbf{b}$ and lies in the plane of $\mathbf{a}$ and $\mathbf{b}$. Find $\mathbf{a}_b$ and $\mathbf{a}_p$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

26. The **law of cosines** for a triangle with sides of length a, b, and c, in which the angle opposite the side of length c is C, takes the form

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Prove this by taking vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that $\mathbf{c} = \mathbf{a} - \mathbf{b}$ and considering the dot product $\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$.

27. The work units W done by a constant force $\mathbf{F}$ when moving its point of application along a straight line L parallel to a vector $\mathbf{a}$ are defined as the product of the component of $\mathbf{F}$ in the direction of $\mathbf{a}$ and the distance d moved along line L. Express W in terms of $\mathbf{F}$, $\mathbf{a}$, and d.

28. If $\mathbf{a}$ and $\mathbf{b}$ are arbitrary vectors and λ and μ are any two scalars, prove that

$$\|\lambda\mathbf{a} + \mu\mathbf{b}\|^2 \le \lambda^2\|\mathbf{a}\|^2 + 2\lambda\mu\mathbf{a} \cdot \mathbf{b} + \mu^2\|\mathbf{b}\|^2.$$

29. Verify the result of Exercise 28 by setting $\lambda = 2, \mu = -3$, $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

2.3 The Cross Product

A product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can be defined in such a way that the result is a vector. The result is written $\mathbf{a} \times \mathbf{b}$ and called the **cross product** of $\mathbf{a}$ and $\mathbf{b}$. The name **vector product** is also used in place of the term *cross product*.

Before defining the cross product we first formulate what is called the right-hand rule. Given any two skew vectors $\mathbf{a}$ and $\mathbf{b}$, the right-hand rule is used to determine

the sense of a third vector **c** that is required to be normal to the plane containing vectors **a** and **b**.

right-hand rule

The Right-Hand Rule

Let **a** and **b** be two arbitrary skew vectors with the same base point, with **c** a vector normal to the plane containing them. If the fingers of the right hand are curled in such a way that they point from vector **a** to vector **b** through the angle θ between them, with $0 < \theta < \pi$, then when the thumb is extended away from the palm it will point in the direction of vector **c**.

When applying the right-hand rule, the order of the vectors is important. If vectors **a**, **b**, and **c** obey the right-hand rule, they will always be written in the order **a**, **b**, **c**, with the understanding that **c** is normal to the plane of **a** and **b**, with its sense determined by the right-hand rule. Figure 2.18 illustrates the right-hand rule.

An important special case of the right-hand rule has already been encountered in connection with the unit vectors **i**, **j**, and **k** that obey the rule, and because the vectors are mutually orthogonal the vectors **j**, **k**, **i** and **k**, **i**, **j** also obey the right-hand rule.

geometrical definition of a cross product

The cross product (a geometrical interpretation)

Let **a** and **b** be two arbitrary vectors, with $\hat{\mathbf{n}}$ a unit vector normal to the plane of **a** and **b** chosen so that **a**, **b**, and $\hat{\mathbf{n}}$, in this order, obey the right-hand rule. Then the **cross product** of vectors **a** and **b**, written $\mathbf{a} \times \mathbf{b}$, is defined as the vector

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|.\|\mathbf{b}\| \sin\theta\,\hat{\mathbf{n}}. \tag{34}$$

This geometrical definition of the cross product is useful in many situations, but when the vectors **a** and **b** are specified in terms of their cartesian components a different form of the definition will be needed.

The cross product can be interpreted as a *vector area*, in the sense that it can be written $\mathbf{a} \times \mathbf{b} = S\hat{\mathbf{n}}$, where $S = OA \cdot BN = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \sin\theta$ is the geometrical area

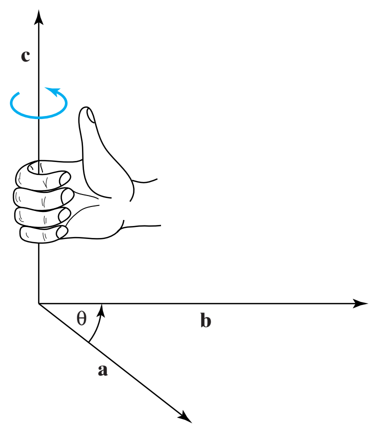


FIGURE 2.18 The right-hand rule.

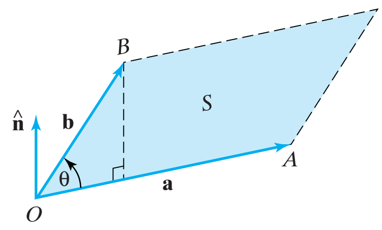


FIGURE 2.19 The cross product interpreted as the vector area of a parallelogram.

of the parallelogram in Fig. 2.19, and the unit vector $\hat{\mathbf{n}}$ is normal to the area. This shows that the geometrical area S of the vector parallelogram with sides **a** and **b** is simply the modulus of the cross product $\mathbf{a} \times \mathbf{b}$, so $S = \|\mathbf{a} \times \mathbf{b}\|$.

properties of the cross product

Properties of the cross product

The following results are consequences of the definition of the cross product.

The cross product is anticommutative

$$\boxed{\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}} \tag{35}$$

The cross product is associative

$$\boxed{\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.} \tag{36}$$

Parallel vectors ($\theta = 0$)

If vectors **a** and **b** are parallel, then

$$\boxed{\mathbf{a} \times \mathbf{b} = \mathbf{0}.} \tag{37}$$

Orthogonal vectors ($\theta = \pi/2$)

If vectors **a** and **b** are orthogonal, then

$$\boxed{\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|.\|\mathbf{b}\|\hat{\mathbf{n}}.} \tag{38}$$

Product of unit vectors

If **a** and **b** are unit vectors, then

$$\boxed{\mathbf{a} \times \mathbf{b} = \sin\theta\,\hat{\mathbf{n}}.} \tag{39}$$

An immediate consequence of properties (34), (35), and (37) is that

$$\boxed{\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},} \tag{40}$$